Boosting Frank-Wolfe by Chasing Gradients

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37th International Conference on Machine Learning
July 12–18, 2020

Outline

Introduction

Let $\mathcal{H}$ be a Euclidean space (e.g., $\mathbb{R}^n$ or $\mathbb{R}^{m\times n}$) and consider

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } x \in \mathcal{C} \end{aligned}$$

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Example

- Sparse logistic regression

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} \frac{1}{m}\sum_{i=1}^{m} \ln(1+\exp(-y_i a_i^\top x)) \\ &\text{s.t. } \|x\|_1 \leqslant \tau \end{aligned}$$

- Low-rank matrix completion

$$\begin{aligned} &\min_{X\in\mathbb{R}^{m\times n}} \frac{1}{2|\mathcal{I}|}\sum_{(i,j)\in\mathcal{I}} (Y_{i,j} - X_{i,j})^2 \\ &\text{s.t. } \|X\|_{\text{nuc}} \leqslant \tau \end{aligned}$$

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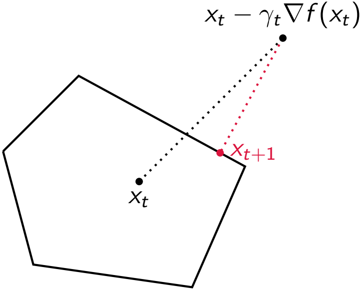

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Feasible region $\mathcal{C}$	**Linear minimization**	**Projection**
$\ell_1/\ell_2/\ell_\infty$-ball	$\mathcal{O}(n)$	$\mathcal{O}(n)$
ℓ_p-ball, $p \in]1, \infty[\setminus \{2\}$	$\mathcal{O}(n)$	N/A
Nuclear norm-ball	$\mathcal{O}(\text{nnz})$	$\mathcal{O}(mn \min\{m, n\})$
Flow polytope	$\mathcal{O}(n)$	$\mathcal{O}(n^{3.5})$
Birkhoff polytope	$\mathcal{O}(n^3)$	N/A
Matroid polytope	$\mathcal{O}(n \ln(n))$	$\mathcal{O}(\text{poly}(n))$

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- Can we avoid projections?

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Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

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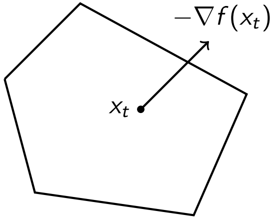

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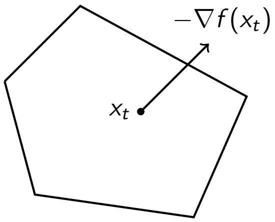

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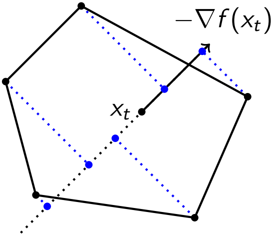

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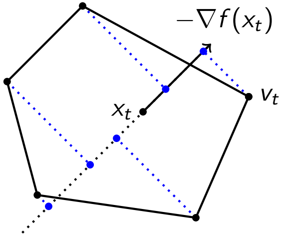

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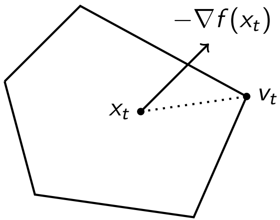

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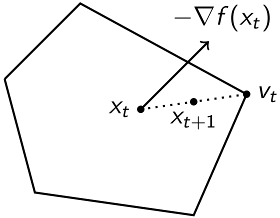

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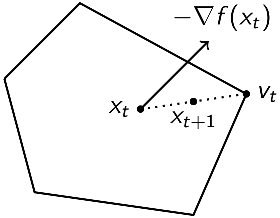

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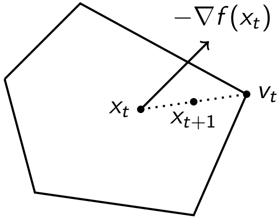

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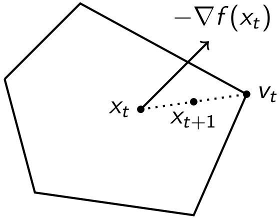

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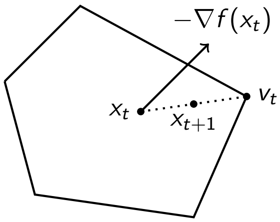

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- Successfully applied to: traffic assignment, computer vision, optimal transport, adversarial learning, etc.

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Theorem (Levitin & Polyak, 1966; Jaggi, 2013)

Let $\mathcal{C} \subset \mathcal{H}$ *be a compact convex set with diameter* D *and* $f : \mathcal{H} \to \mathbb{R}$ *be a* L*-smooth convex function, and let* $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ *for some* $y \in \mathcal{C}$*. If* $\gamma_t = \frac{2}{t+2}$ *(default) or* $\gamma_t = \min\left\{ \frac{\langle \nabla f(x_t), x_t - v_t \rangle}{L\|x_t - v_t\|^2}, 1 \right\}$ *("short step"), then*

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- Why?

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Consider the simple problem

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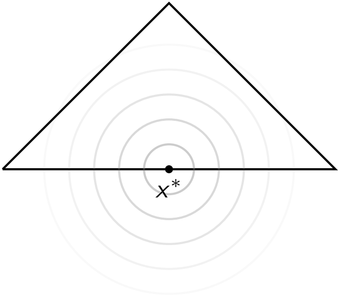
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



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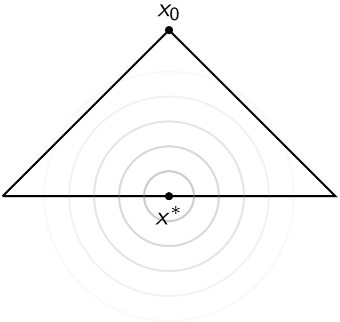

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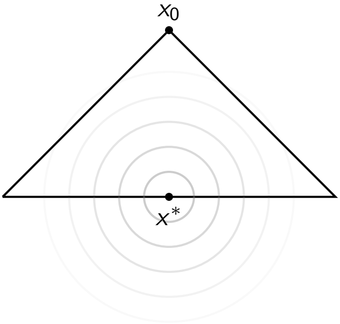

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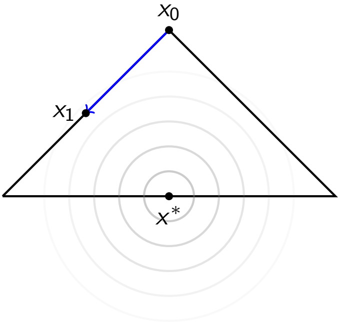

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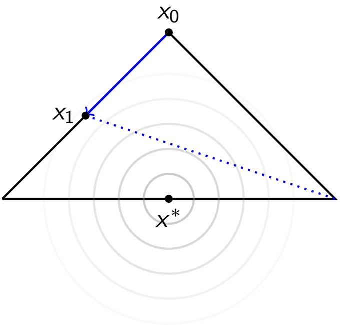

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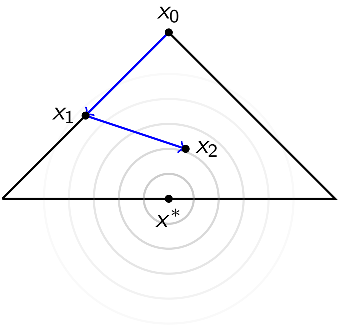

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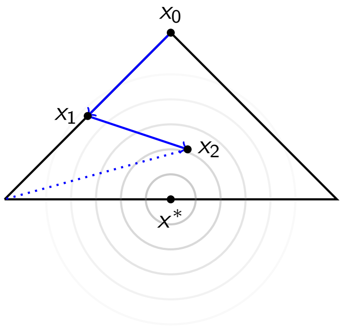

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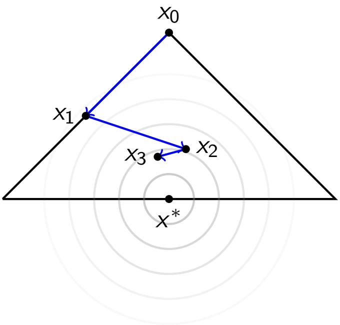

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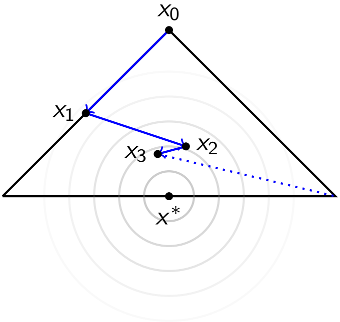

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The Frank-Wolfe algorithm

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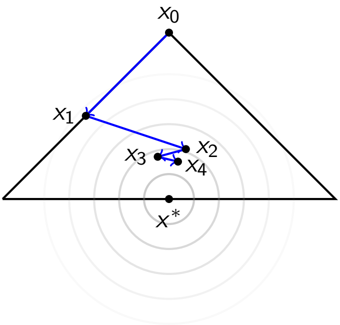

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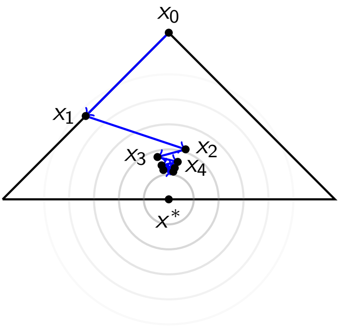

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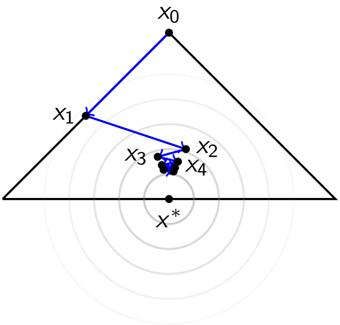

- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices
- This yields an inefficient zig-zagging trajectory

Improved Frank-Wolfe variants

- Away-Step Frank-Wolfe (AFW) (Wolfe, 1970; Lacoste-Julien & Jaggi, 2015): enhances FW by allowing to move away from vertices

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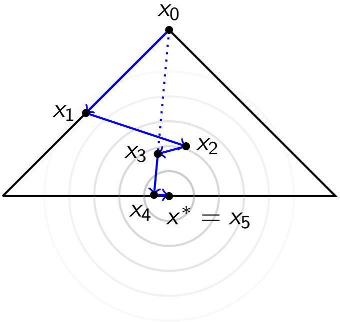

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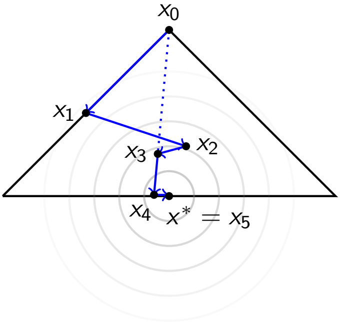

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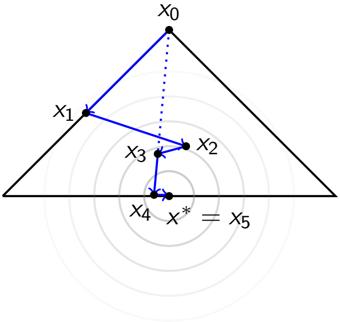

- Decomposition-Invariant Pairwise Conditional Gradient (DICG) (Garber & Meshi, 2016): memory-free variant of AFW
- Blended Conditional Gradients (BCG) (Braun et al., 2019): blends FCFW and FW

Boosting Frank-Wolfe

- Can we speed up FW in a simple way?

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Idea:

- Speed up FW by moving in a direction better aligned with $-\nabla f(x_t)$
- Build this direction by using $\mathcal{V}$ to maintain the projection-free property

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- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

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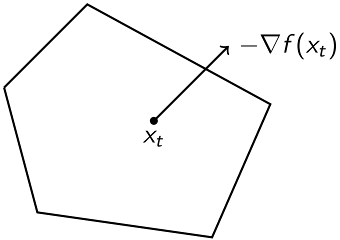

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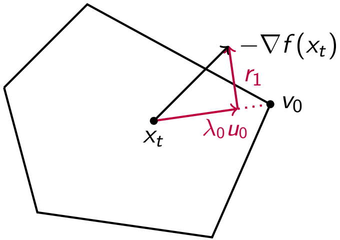

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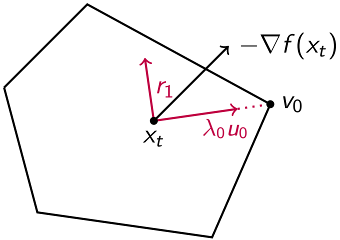

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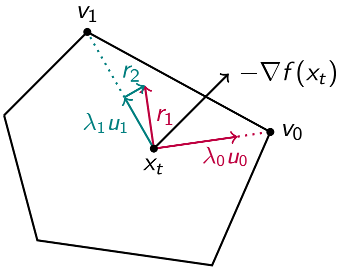

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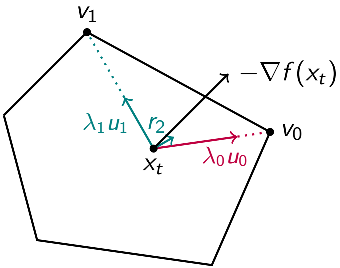

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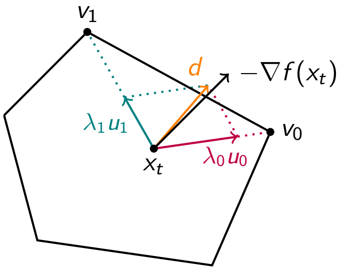

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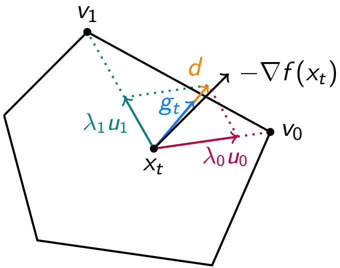

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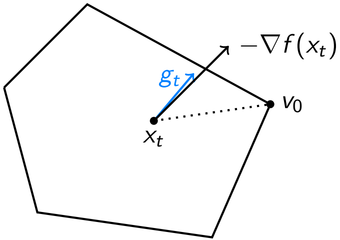

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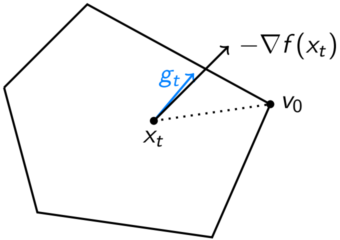

- The boosted direction g_t is better aligned with $-\nabla f(x_t)$ than is the FW direction $v_0 - x_t$ and satisfies $[x_t, x_t + g_t] \subseteq \mathcal{C}$ so we can update

$$x_{t+1} = x_t + \gamma_t g_t \quad \text{for any } \gamma_t \in [0, 1]$$

Boosting Frank-Wolfe

Why $[x_t, x_t + g_t] \subseteq \mathcal{C}$? Let K_t be the number of alignment rounds. We have

$$d = \sum_{k=0}^{K_t-1} \lambda_k (v_k - x_t) \quad \text{where } \lambda_k > 0 \text{ and } v_k \in \mathcal{V}$$

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Thus, $x_t + g_t \in \mathcal{C}$ so $[x_t, x_t + g_t] \subseteq \mathcal{C}$ by convexity

Boosting Frank-Wolfe

Algorithm Finding a direction g *well* aligned with ∇ from a reference point z

Input: $z \in \mathcal{C}$, $\nabla \in \mathcal{H}$, $K \in \mathbb{N}\backslash\{0\}$, $\delta \in]0,1[$.

```
d_0 ← 0, Λ ← 0
for k = 0 to K − 1 do
    r_k ← ∇ − d_k                                         ▷ k-th residual
    v_k ← arg max_{v∈𝒱} ⟨r_k, v⟩                           ▷ FW oracle
    u_k ← arg max_{u∈{v_k − z, −d_k/‖d_k‖}} ⟨r_k, u⟩
    λ_k ← ⟨r_k, u_k⟩/‖u_k‖²
    d'_k ← d_k + λ_k u_k
    if align(∇, d'_k) − align(∇, d_k) ⩾ δ then
        d_{k+1} ← d'_k
        Λ_t ← { Λ + λ_k               if u_k = v_k − z
              { Λ(1 − λ_k/‖d_k‖)      if u_k = −d_k/‖d_k‖
    else
        break                                             ▷ exit k-loop
g ← d_k/Λ                                                 ▷ normalization
```

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$d_0 \leftarrow 0$, $\Lambda \leftarrow 0$
for $k = 0$ **to** $K-1$ **do**
  $r_k \leftarrow \nabla - d_k$ ▷ k-th residual
  $v_k \leftarrow \arg\max_{v \in \mathcal{V}} \langle r_k, v \rangle$ ▷ FW oracle
  $u_k \leftarrow \arg\max_{u \in \{v_k - z, -d_k/\|d_k\|\}} \langle r_k, u \rangle$
  $\lambda_k \leftarrow \langle r_k, u_k \rangle / \|u_k\|^2$
  $d'_k \leftarrow d_k + \lambda_k u_k$
  if $\text{align}(\nabla, d'_k) - \text{align}(\nabla, d_k) \geqslant \delta$ **then**
    $d_{k+1} \leftarrow d'_k$
    $\Lambda_t \leftarrow \begin{cases} \Lambda + \lambda_k & \textbf{if } u_k = v_k - z \\ \Lambda(1 - \lambda_k/\|d_k\|) & \textbf{if } u_k = -d_k/\|d_k\| \end{cases}$
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$g \leftarrow d_k/\Lambda$ ▷ normalization

- Technicality to ensure convergence of the procedure (Locatello et al., 2017)

Boosting Frank-Wolfe

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- Technicality to ensure convergence of the procedure (Locatello et al., 2017)
- The stopping criterion is an alignment improvement condition (typically $\delta = 10^{-3}$ and $K = +\infty$)

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Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$

Algorithm Boosted Frank-Wolfe (BoostFW)

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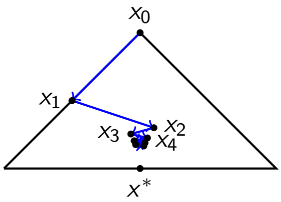

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for $t = 0$ **to** $T-1$ **do**
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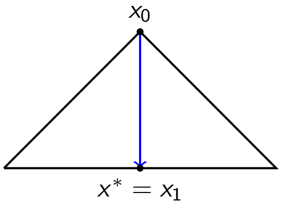

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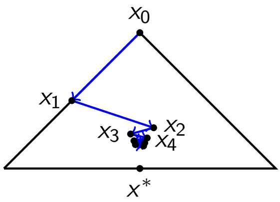

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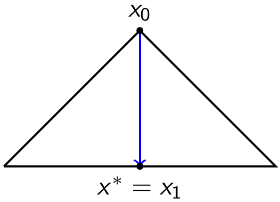

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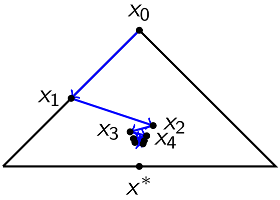

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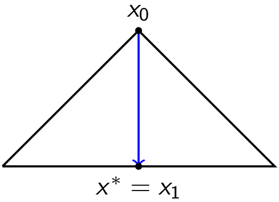

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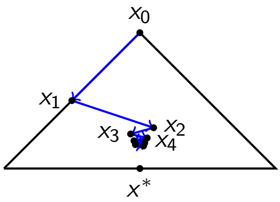

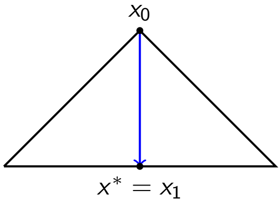

- What is the convergence rate of BoostFW?
- Is BoostFW expensive in practice?
- How does it compare to the state-of-the-art?

Boosting Frank-Wolfe

- Let N_t be the number of iterations up to t where at least 2 rounds of alignment were performed (FW = always 1 round)

Theorem

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth, convex, and μ-gradient dominated function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. Set $\gamma_t = \min\left\{\frac{\langle -\nabla f(x_t), g_t \rangle}{L\|g_t\|^2}, 1\right\}$ ("short step") and suppose that $N_t \geqslant \omega t^p$ where $p \in]0,1]$. Then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{LD^2}{2} \exp\left(-\delta^2 \frac{\mu}{L} \omega t^p\right)$$

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- In practice, $N_t \approx t$ (so $\omega \lesssim 1$ and $p = 1$)

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$$\begin{aligned}
\min_{x\in\mathbb{R}^{|\mathcal{A}|}} \ & \sum_{a\in\mathcal{A}} \tau_a x_a \left(1 + 0.03\left(\frac{x_a}{c_a}\right)^4\right) \\
\text{s.t. } & x_a = \sum_{r\in\mathcal{R}} \mathbb{1}_{\{a\in r\}} y_r && a\in\mathcal{A} \\
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\end{aligned}$$

$$\min_{x\in\mathbb{R}^n} \frac{1}{m}\sum_{i=1}^m \ln(1+\exp(-y_i a_i^\top x)) \quad \text{s.t. } \|x\|_1 \leqslant \tau$$

$$\min_{X\in\mathbb{R}^{m\times n}} \frac{1}{|\mathcal{I}|}\sum_{(i,j)\in\mathcal{I}} h_\rho(Y_{i,j} - X_{i,j}) \quad \text{s.t. } \|X\|_{\text{nuc}} \leqslant \tau$$

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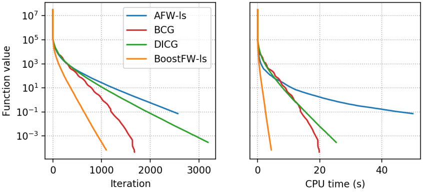
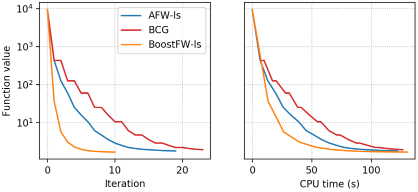
- For BoostFW and AFW we also run the line search-free variants (the "short step" strategy) and label them with an "L"

Computational experiments

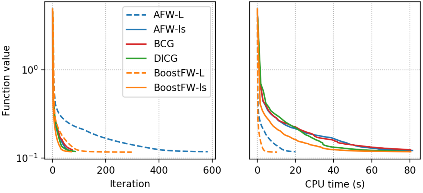
- Sparse signal recovery



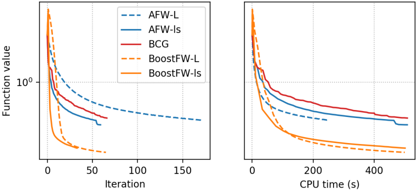
- Traffic assignment



- Sparse logistic regression on the Gisette dataset



- Collaborative filtering on the MovieLens 100k dataset



Boosting DICG

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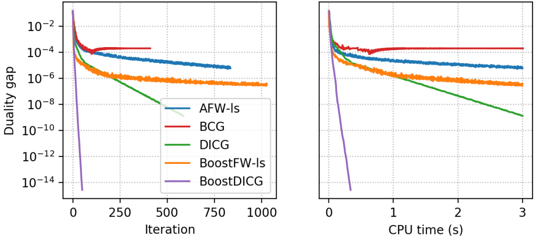

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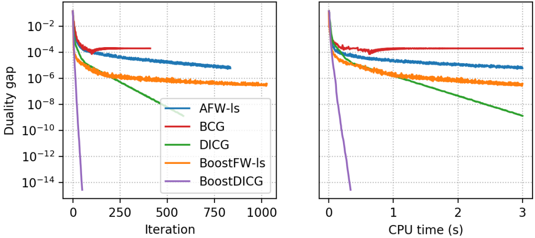

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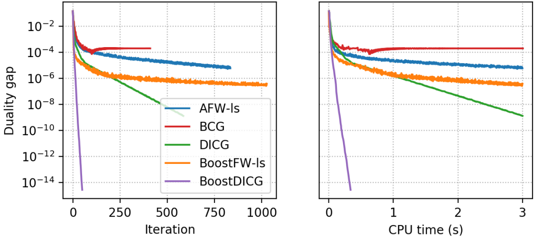

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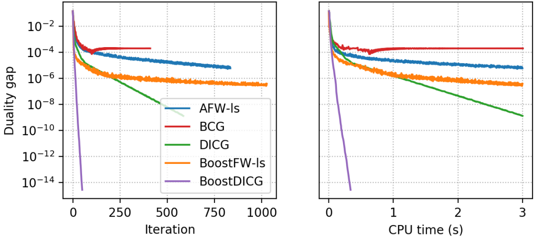

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  E.g., large-scale finite-sum/stochastic constrained optimization:

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g_t &\leftarrow \text{procedure}(x_t, -\tilde{\nabla} f(x_t), K, \delta) \\
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