Amortized Finite Element Analysis for Fast PDE-Constrained Optimization

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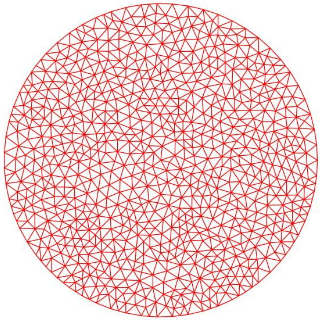
The Finite Element Analysis (FEA) in a Traditional Flow

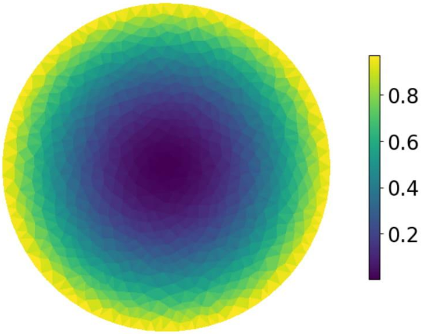
Heat problem:

$$
\begin{aligned}
-\Delta u &= \lambda \quad \text{in } \Omega, \\
u &= u_b \quad \text{on } \Gamma,
\end{aligned}
$$

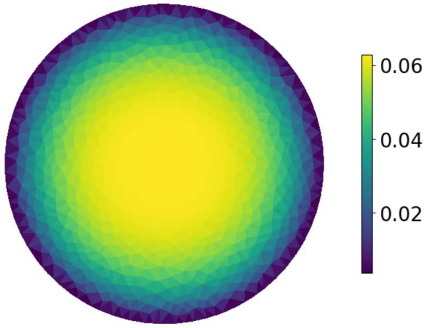
λ
source
(control parameter) → **FEA** → u
solution
(state variable)



FEA mesh

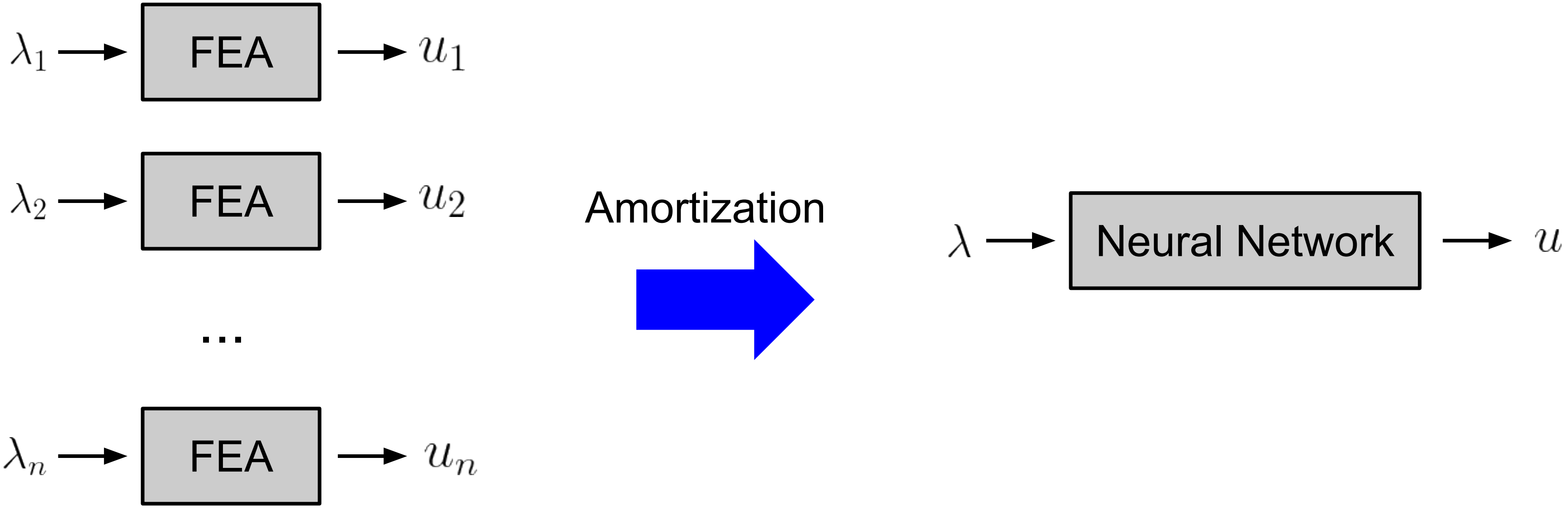


source (control parameter)



solution (state variable)

Can we learn from FEA?



The FEA Road Map

$$\begin{array}{ccc} \textit{Strong Formulation} & & \\ \Downarrow & & \\ \textit{Weak Formulation} & \approx & \textit{Galerkin Weak Formulation} \\ \Updownarrow & & \Updownarrow \\ \textit{Minimization Formulation} & \approx & \boxed{\textit{Galerkin Minimization Formulation}} \end{array}$$

Amortized Finite Element Analysis (AmorFEA)

$$\min_{\boldsymbol{u}\in\mathbb{R}^n} \mathcal{L}(\boldsymbol{u}, \boldsymbol{\lambda})$$

FEA: per-control-vector optimization

Amortization

$$\min_{\psi} \mathbb{E}_{p(\boldsymbol{\lambda})}[\mathcal{L}(g_\psi(\boldsymbol{\lambda}), \boldsymbol{\lambda})]$$

AmorFEA: shared regression problem

$g_\psi : \mathbb{R}^m \to \mathbb{R}^n$ A neural network function parametrized by ψ

Connection to Amortized Variational Inference

	Variational Inference	Finite Element Analysis
Functional to minimize	KL divergence	potential energy
Approximate functions	Variational family of distributions	FEA basis functions

Both are variational procedures...

Connection to Amortized Variational Inference

	Amortized Variational Inference[1]	AmorFEA
Input	Observation data points	Control parameters
Output	Variational family parameters	Solutions

Both optimize over neural network parameters...

[1] Kingma & Welling, 2013; Rezende et al., 2014

Amortization Gap

Comments:

1. Fast solver, jumping to the solution from parameter directly
2. Easy to train, no need of (expensive) supervised data
3. Only advantageous when problems need to be solved repeatedly
4. Induced error: Amortization gap (also see [1])

$$\mathbb{E}_{p(\boldsymbol{\lambda})}[\mathcal{L}(g_{\psi^\star}(\boldsymbol{\lambda}), \boldsymbol{\lambda})]$$

Amortization gap

$$\mathbb{E}_{p(\boldsymbol{\lambda})}[\mathcal{L}(\boldsymbol{u}^\star_{\boldsymbol{\lambda}}, \boldsymbol{\lambda})]$$

Approximation gap

$$\mathbb{E}_{p(\boldsymbol{\lambda})}[\mathcal{L}(u^e, \boldsymbol{\lambda})]$$

[1] Cremer et al. (2018)

Deployment of AmorFEA in PDE-constrained Optimization

Discretized PDE-constrained optimization

$$\min_{\boldsymbol{u}\in\mathbb{R}^n,\boldsymbol{\lambda}\in\mathbb{R}^m} \mathcal{J}(\boldsymbol{u},\boldsymbol{\lambda})$$

$$\text{s.t.} \quad \boldsymbol{c}(\boldsymbol{u},\boldsymbol{\lambda}) = 0,$$

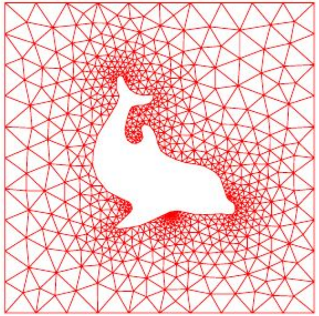
where $\mathcal{J}(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the objective function

$\boldsymbol{c}(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the constraint function imposed by the governing PDE

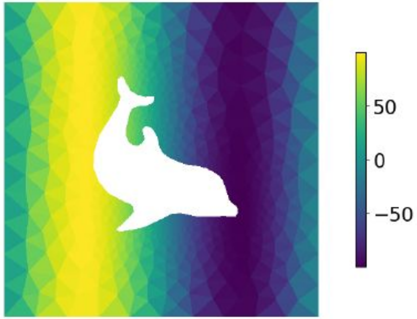
Source Field Finding

Minimize $$\mathcal{J}(u,\lambda) = \frac{1}{2}\int_{\Omega}(u-u_d)^2 dx + \frac{\alpha}{2}\int_{\Omega}\lambda^2 dx$$

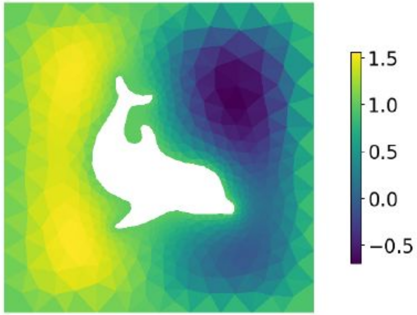
Subject to $$\begin{aligned} -\Delta u + 10(u+u^3) &= \lambda \quad \text{in } \Omega, \\ u &= u_b \quad \text{on } \Gamma, \end{aligned}$$



FEA mesh



source (control parameter)



solution (state variable)

Source Field Finding

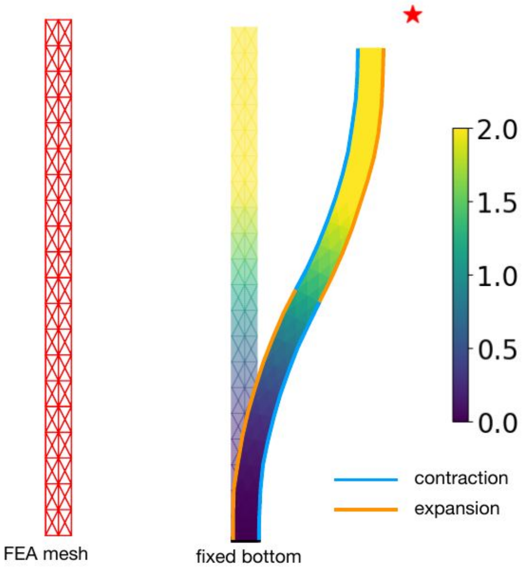
Compare with the adjoint method

Method	α	Optimized objective	Wall time [ms]	Iteration steps [#]
Adjoint Method	10^{-6}	3.35×10^{-4}	700	13
	10^{-3}	2.47×10^{-3}	600	11
	1	2.56×10^{-3}	337	6
AmorFEA	10^{-6}	3.40×10^{-4}	38	13
	10^{-3}	2.47×10^{-3}	37	11
	1	2.56×10^{-3}	16	7

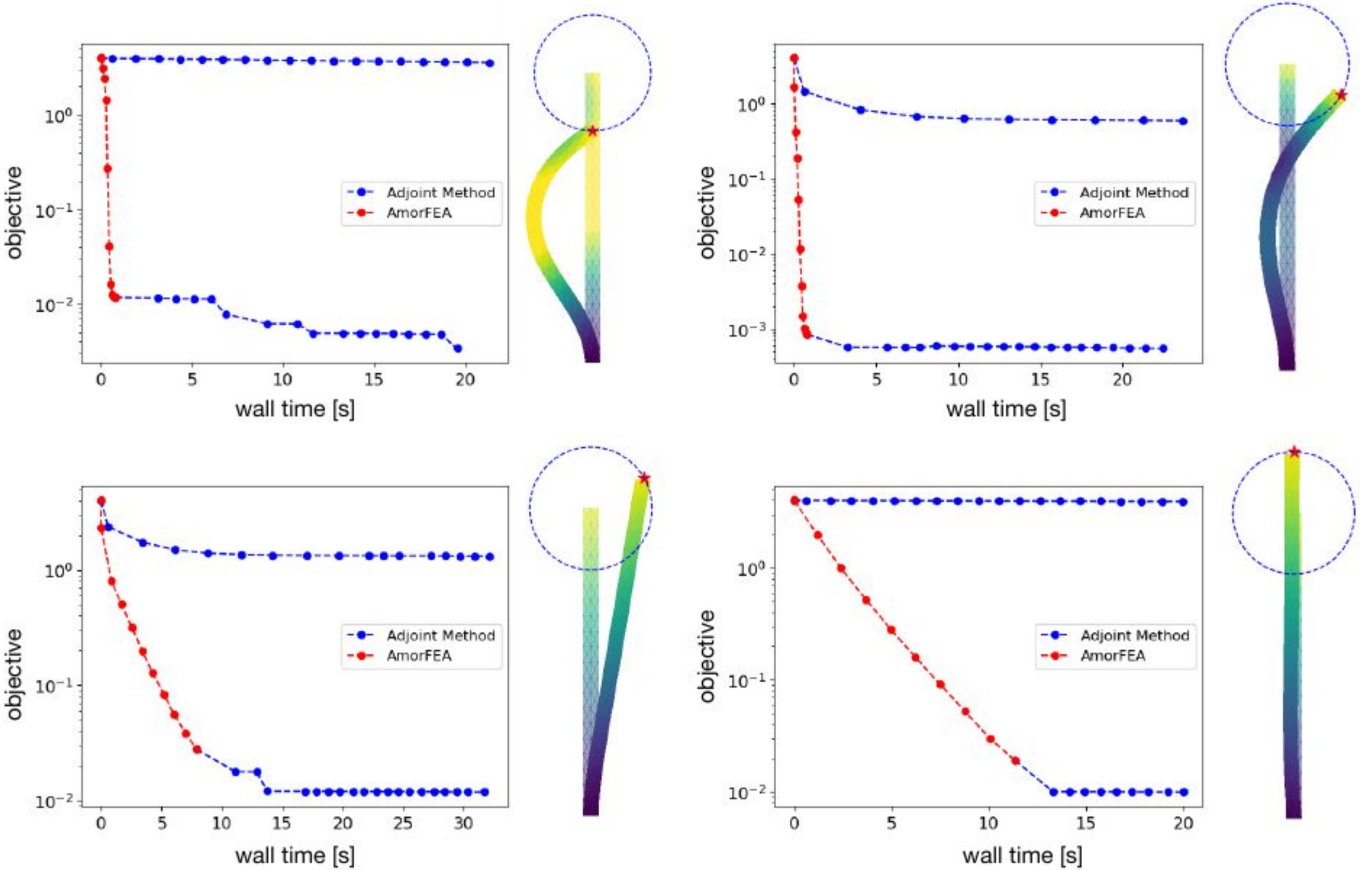
Inverse Kinematics of a Soft Robot

Minimize $\mathcal{J}(u, \lambda) = \|u(x_0) - u_0)\|_2^2$

Subject to
$$\begin{aligned} \text{Div } P(u) &= 0 \quad \text{in } \Omega, \\ r(u, \lambda) &= 0 \quad \text{on } \Gamma, \end{aligned}$$



Inverse Kinematics of a Soft Robot



Thank you