Almost surely constrained convex optimization

Ahmet Alacaoglu
ahmet.alacaoglu@epfl.ch

Laboratory for Information and Inference Systems (LIONS)
École Polytechnique Fédérale de Lausanne (EPFL)
Switzerland

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Joint work with
Olivier Fercoq @ Telecom Paris-Tech, Ion Necoara @ UPB
Volkan Cevher @ LIONS

lions@epfl

EPFL

Problem template

Almost surely constrained convex optimization:

$$\min_{x\in\mathbb{R}^d} \{P(x) := F(x) + h(x)\}$$
$$A(\xi)x \in b(\xi) \quad \xi\text{-almost surely,}$$

- $F(x) = \mathbb{E}\left[f(x,\xi)\right]$, with convex and smooth $f(\cdot,\xi) : \mathbb{R}^d \to \mathbb{R}$.
- $h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a nonsmooth, proximable convex function.
- $A(\xi) \in \mathbb{R}^{m\times d}$ and $b(\xi) \subseteq \mathbb{R}^m$ are random.

• Some applications: support vector machines, basis pursuit, portfolio optimization, semi-infinite programming...

Prior art

$$\boxed{\min_{x\in\mathbb{R}^d} \mathbb{E}\left[f(x,\xi)\right] : \quad x \in \mathcal{B}(:= \cap_{\xi\in\Omega}\mathcal{B}(\xi))}$$

- Idea: Use alternating projections for the constraints.
- Update:

$$\begin{aligned} y^k &= \mathrm{prox}_{\mu_k f(\cdot,\xi)}\left(x^k\right) \\ x^{k+1} &= \mathrm{proj}_{\mathcal{B}(\xi)}(y^k) \end{aligned}$$

- Drawbacks:
  - More restricted problem class.
  - Requires projectability of sets.

Patrascu, A., and Necoara I., "Nonasymptotic convergence of stochastic proximal point methods for constrained convex optimization", JMLR, 2017.

Primer on stochastic proximal gradient method (SPG)

$$\min_{x\in\mathbb{R}^d} \{P(x) := F(x) + h(x)\}$$

- SPG:

$$x^{k+1} = \operatorname{prox}_{\frac{\alpha_0}{\sqrt{k}}h}\left(x^k - \frac{\alpha_0}{\sqrt{k}}\nabla f(x^k,\xi)\right).$$

- Convergence rate:

$$P(x^k) - P(x_\star) \leq \mathcal{O}\left(\frac{\sigma^2 + L\|x^0 - x_\star\|^2}{\sqrt{k}}\right).$$

◦ Standard assumption: Bounded variance:

$$\mathbb{E}\|\nabla F(x) - \nabla f(x,\xi)\|^2 \leq \sigma^2 < \infty.$$

Primer on smoothing

- A smooth estimate of $g = \delta_{b(\xi)}$:

$$g_\beta(A(\xi)x, \xi) = \max_{y \in \mathbb{R}^m} \left\{ \langle A(\xi)x, y \rangle - g^*(y, \xi) - \frac{\beta}{2} \|y\|^2 \right\}.$$

- g_β is differentiable and ∇g_β is $\frac{1}{\beta}$-Lipschitz continuous.

$$g_\beta(A(\xi)x, \xi) = \frac{1}{2\beta} \mathrm{dist}(A(\xi)x, b(\xi))^2$$

$$G_\beta(Ax) = \frac{1}{2\beta} \mathbb{E} \left[\mathrm{dist}(A(\xi)x, b(\xi))^2 \right],$$

where $\mathrm{dist}(x, \mathcal{K}) = \inf_{y \in \mathcal{K}} \|x - y\|$.

Stochastic gradients of smoothed function

Algorithmic Idea: Apply SGD to

$$\min_{x\in\mathbb{R}^d} \left\{ P_\beta(x) := \mathbb{E} f(x,\xi) + h(x) + G_\beta(Ax) \right\},$$

with β decreasing to 0.

- Recall:

$$g_\beta(A(\xi)x,\xi) = \frac{1}{2\beta}\mathrm{dist}(A(\xi)x, b(\xi))^2$$
$$G_\beta(Ax) = \frac{1}{2\beta}\mathbb{E}\left[\mathrm{dist}(A(\xi)x, b(\xi))^2\right].$$

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- Recall:

$$g_\beta(A(\xi)x,\xi) = \frac{1}{2\beta}\mathrm{dist}(A(\xi)x, b(\xi))^2$$
$$G_\beta(Ax) = \frac{1}{2\beta}\mathbb{E}\left[\mathrm{dist}(A(\xi)x, b(\xi))^2\right].$$

- Taking stochastic gradients:

$$\nabla_x g_\beta(A(\xi)x,\xi) = A(\xi)^\top \nabla_{A(\xi)x}\frac{1}{2\beta}\mathrm{dist}(A(\xi)x, b(\xi))^2$$
$$= \frac{1}{\beta}A(\xi)^\top(A(\xi)x - \mathrm{proj}_{b(\xi)}(A(\xi)x)).$$

- Only requires projections to $b(\xi)$.
- Challenge: Standard variance bound does not hold as $\beta \to 0$.

SASC for general convex case

Input: $x_0^0 \in \mathbb{R}^d$

Parameters: $\alpha_0 \leq \dfrac{3}{4L(\nabla F)}$, and $\omega > 1$

$m_0 \in \mathbb{N}_*$.

for $s \in \mathbb{N}$ **do**

$m_s = \lfloor m_0 \omega^s \rfloor$, and $\alpha_s = \alpha_0 \omega^{-s/2}$.

$\beta_s = 4\alpha_s \sup_\xi \|A(\xi)\|^2$.

for $k \in \{0, \dots, m_s - 1\}$ **do**

Draw $\xi = \xi_{k+1}^s$.

$x_{k+1}^s =$

$\text{prox}_{\alpha_s h}\left(x_k^s - \alpha_s\left[\nabla f(x_k^s, \xi) + \frac{1}{\beta_s} A(\xi)^\top (A(\xi) x_k^s - \text{proj}_{b(\xi)}(A(\xi) x_k^s))\right]\right)$

end for

$\bar{x}^s = \frac{1}{m_s} \sum_{k=1}^{m_s} x_k^s$

$x_0^{s+1} = x_{m_s}^s$.

end for

return $\bar{x}^s$

Lagrangian, primal-dual solutions

$$\min_{x \in \mathbb{R}^d} \{P(x) := F(x) + h(x)\}$$
$$A(\xi)x \in b(\xi) \quad \xi\text{-almost surely,}$$

- Define the Lagrangian:

$$\mathcal{L}(x,y) = P(x) + \int \langle A(\xi)x, y(\xi)\rangle - \mathrm{supp}_{b(\xi)}(y(\xi))\mu(d\xi),$$

where $\mathrm{supp}_{\mathcal{K}}(y) = \sup_{x \in \mathcal{K}} \langle x, y\rangle$.

- $(x_\star, y_\star)$ is a saddle point of

$$\min_{x \in \mathbb{R}^d} \max_{y \in \mathcal{Y}} \mathcal{L}(x,y).$$

A key lemma

$$\min_{x \in \mathbb{R}^d} \{P(x) := F(x) + h(x)\}$$
$$A(\xi)x \in b(\xi) \quad \xi\text{-almost surely,}$$

- Define $S_\beta(x) = P_\beta(x) - P(x_\star) = P(x) - P(x_\star) + \frac{1}{2\beta} \int \operatorname{dist}(A(\xi)x, b(\xi))^2 \mu(d\xi)$.

Then, the following hold:

$$S_\beta(x) \geq -\frac{\beta}{2}\|y_\star\|^2,$$
$$P(x) - P(x_\star) \geq -\frac{1}{4\beta}\int \operatorname{dist}(A(\xi)x, b(\xi))^2 \mu(d\xi) - \beta\|y_\star\|^2,$$
$$P(x) - P(x_\star) \leq S_\beta(x),$$
$$\int \operatorname{dist}(A(\xi)x, b(\xi))^2 \mu(d\xi) \leq 4\beta^2\|y_\star\|^2 + 4\beta S_\beta(x).$$

If S_β and β are small, then objective residual and feasibility values are also small.

Main theorem

$$\min_{x\in\mathbb{R}^d} \{P(x) := F(x) + h(x)\}$$
$$A(\xi)x \in b(\xi) \quad \xi\text{-almost surely,}$$

- Denote by $M_s = \sum_{i=0}^{s} m_i$ total number of iterations. Then, the iterates of SASC satisfy

$$\mathbb{E}|P(\bar{x}^s) - P(x_\star)| \le \mathcal{O}\left(\log_\omega(M_s/m_0)\frac{\sigma_f^2 + \|x_\star - x_0^0\|^2 + \|y_\star\|^2}{\sqrt{M_s}}\right),$$

$$\sqrt{\mathbb{E}\left[\mathrm{dist}(A(\xi)\bar{x}^s, b(\xi))^2\right]} \le \mathcal{O}\left(\log_\omega(M_s/m_0)\frac{\sigma_f^2 + \|x_\star - x_0^0\|^2 + \|y_\star\|^2}{\sqrt{M_s}}\right).$$

- This rate is optimal even without constraints up to a logarithmic factor.

Extensions: Restricted strongly convex

$$\min_{x\in\mathbb{R}^d} \{P(x) := F(x) + h(x)\}$$
$$A(\xi)x \in b(\xi) \quad \xi\text{-almost surely,}$$

- Denote by $M_s = \sum_{i=0}^{s} m_i$ total number of iterations.
- If $P(x)$ satisfies the quadratic growth condition:

$$P(x) - P(x_\star) \geq \frac{\mu}{2}\|x - x_\star\|^2,$$

the iterates of SASC satisfy

$$\mathbb{E}|P(\bar{x}^s) - P(x_\star)| \leq \mathcal{O}\left(\log_\omega(M_s/m_0)\frac{\sigma_f^2 + \|x_\star - x_0^0\|^2 + \|y_\star\|^2}{M_s}\right),$$
$$\sqrt{\mathbb{E}\left[\text{dist}(A(\xi)\bar{x}^s, b(\xi))^2\right]} \leq \mathcal{O}\left(\log_\omega(M_s/m_0)\frac{\sigma_f^2 + \|x_\star - x_0^0\|^2 + \|y_\star\|^2}{M_s}\right).$$

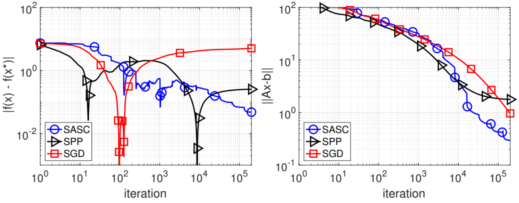
- This rate is optimal even without constraints up to a logarithmic factor.

Numerical experiments: Basis pursuit

$$\min_{x \in \mathbb{R}^d} \|x\|_1$$
$$\text{st: } a^\top x = b, a.s.$$

- Data generation:
  - $\Sigma_{i,j} = \rho^{|i-j|}$ with $\rho = 0.9$.
  - $x^* \in \mathbb{R}^d$, $d = 100$ with 10 nonzero coefficients.
  - $a_i \sim \mathcal{N}(0, \Sigma)$ independent random variables, which are then centered and normalized.
  - $b_i = a_i^\top x^*$, $i \in [1, m]$ where $m = 10^5$.
- Because of the centering, there are multiple solutions to the infinite system $a^\top x = b$ a.s.

Numerical experiments: Basis pursuit



- SGD does not converge to the sparse solution.
- SPP stagnates at the predefined accuracy, due to fixed step size.

Patrascu, A., and Necoara I., "Nonasymptotic convergence of stochastic proximal point methods for constrained convex optimization", JMLR, 2017.

Numerical experiments: SVM

Hard margin SVM:

$$\min_{x \in \mathbb{R}^d} \frac{1}{2}\|x\|^2 : b_i \langle a_i, x \rangle \geq 1, \forall i.$$

- SASC applies to hard margin SVM.

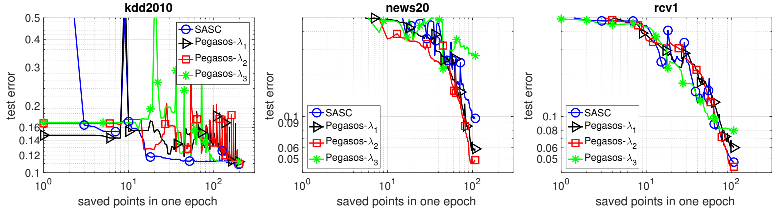
Soft margin SVM:

$$\min_{x \in \mathbb{R}^d} \frac{1}{2}\|x\|^2 + C \sum_{i=1}^{n} \max \{0, 1 - b_i \langle a_i, x \rangle\},$$

- Pegasos (primal subgradient method) applies to soft margin SVM.

Shalev-Shwartz, S, et al. "Pegasos: Primal estimated sub-gradient solver for svm." Math. Prog., 2011

Numerical experiments: SVM



Dataset 1: kdd2010: 19,264,997 training examples, 748,401 testing examples, 1,163,024 features

Dataset 2: news20: 17,996 training examples, 2,000 testing examples, 1,355.191 features

Dataset 3: rcv1: 20,424 training examples, 677, 399 testing examples, 47,236 features

- Accuracy of Pegasos depends on the regularization parameter.
- SASC is comparable to Pegasos with the best regularization parameter.

Shalev-Shwartz, S, et al. "Pegasos: Primal estimated sub-gradient solver for svm." Math. Prog., 2011

Conclusions

• SGD-type method for stochastic optimization with infinitely many linear inclusion constraints.

• Optimal convergence rates upto a logarithmic factor.

• Extensions for solving

$$\min_{x \in \mathbb{R}^d} \mathbb{E}\left[f(x,\xi) + g_1(A_1(\xi)x, \xi)\right] + h(x), \quad A_2(\xi)x \in b(\xi), \xi\text{-almost surely,}$$

with nonsmooth and Lipschitz continuous g_1.

• State-of-the-art practical performance.

To learn more: `ahmet.alacaoglu@epfl.ch`

• Poster @ Pacific Ballroom #101