Improved Convergence for ℓ_∞ and ℓ_1 Regression via Iteratively Reweighted Least Squares

Alina Ene, Adrian Vladu

BOSTON UNIVERSITY

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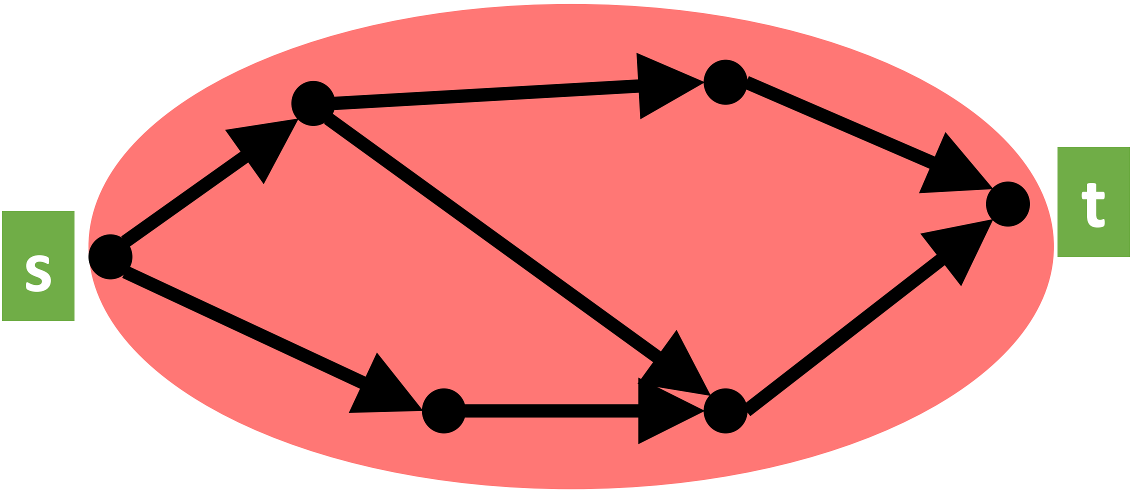
Benchmark: Optimization on Graphs



$$\min |\mathbf{x}|_\infty$$
$$A\mathbf{x} = b$$

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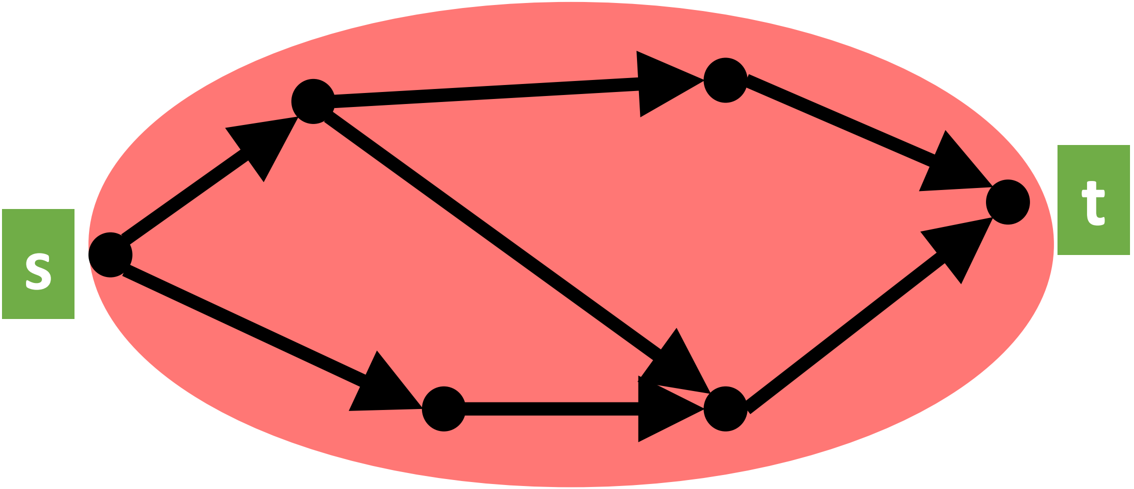

minimize congestion of flow **x**

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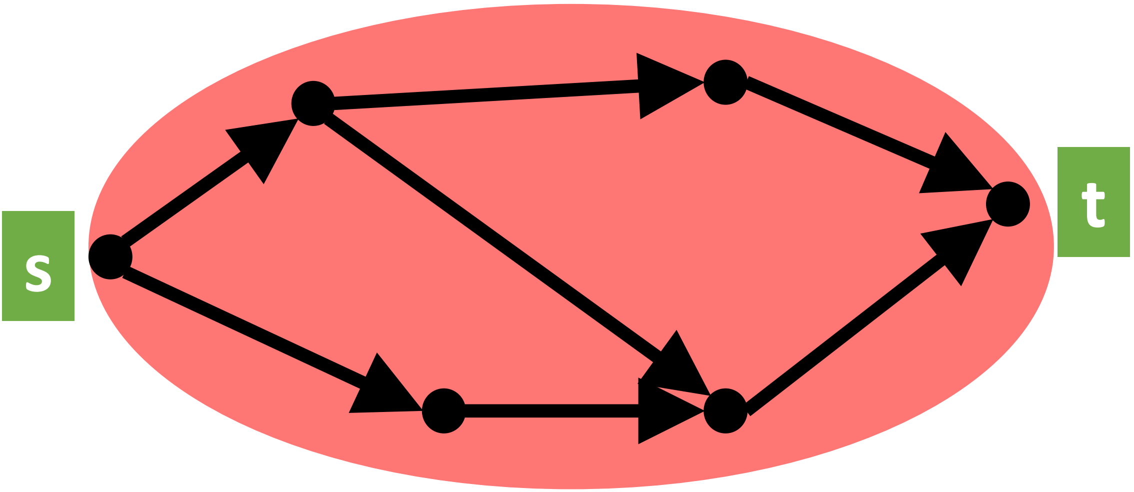

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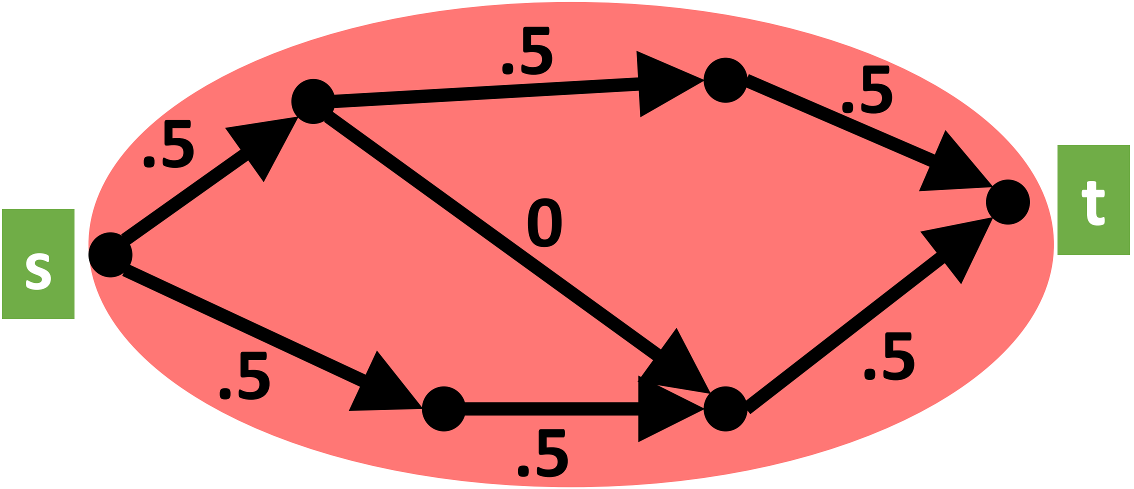

Maximum flow

minimize congestion of flow x

$$\min |x|_\infty$$
$$Ax = b$$

boundary condition: x routes demand from s to t

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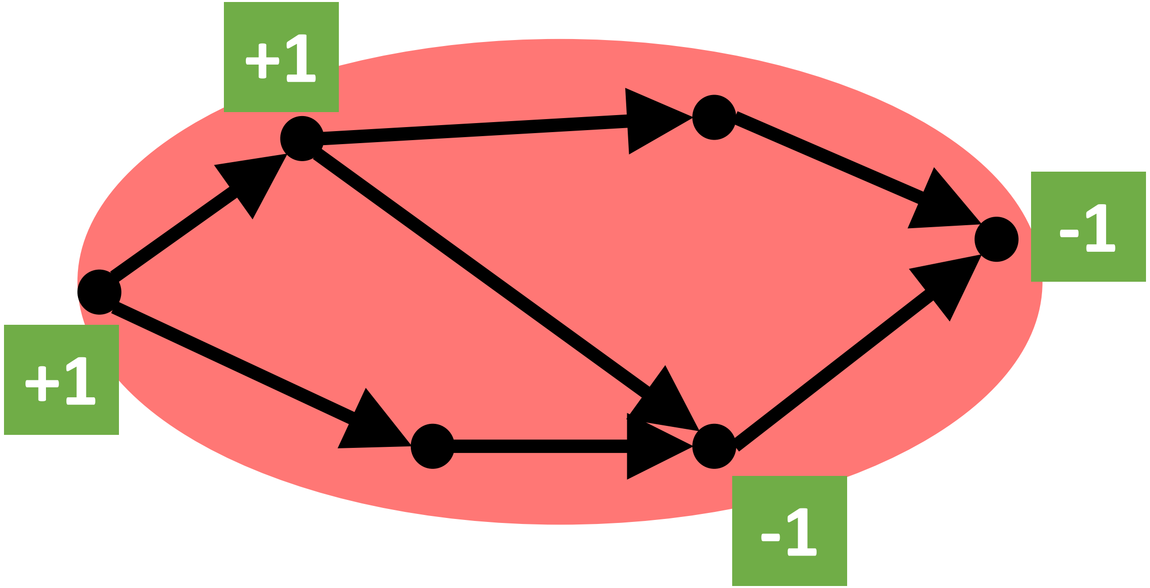

$$\min \ |\mathbf{x}|_1$$
$$A\mathbf{x} = \mathbf{b}$$

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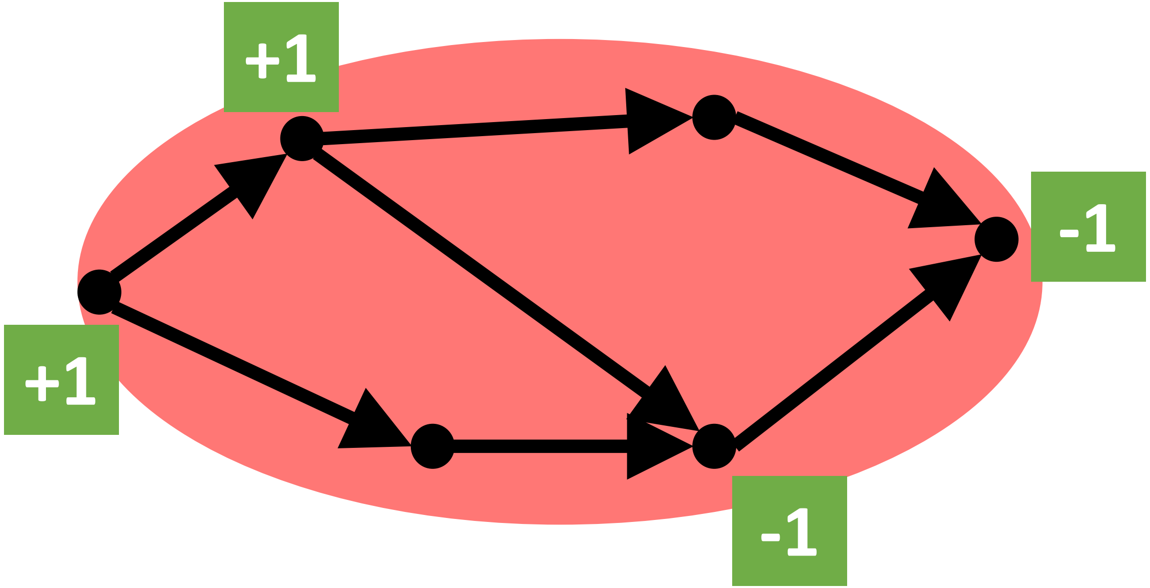

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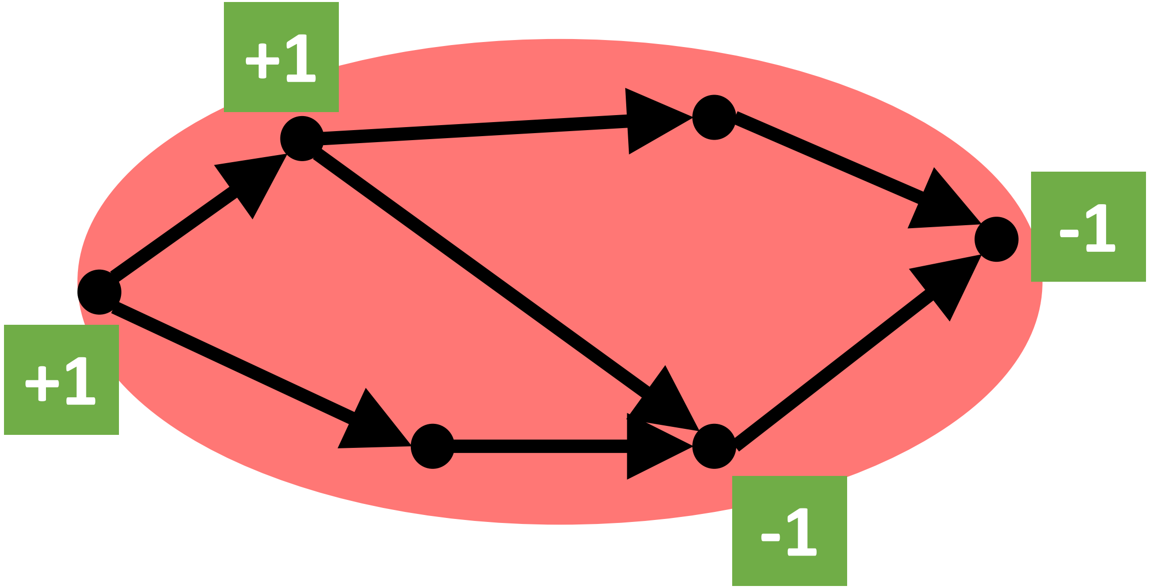

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$\mathbf{x}$ routes demand
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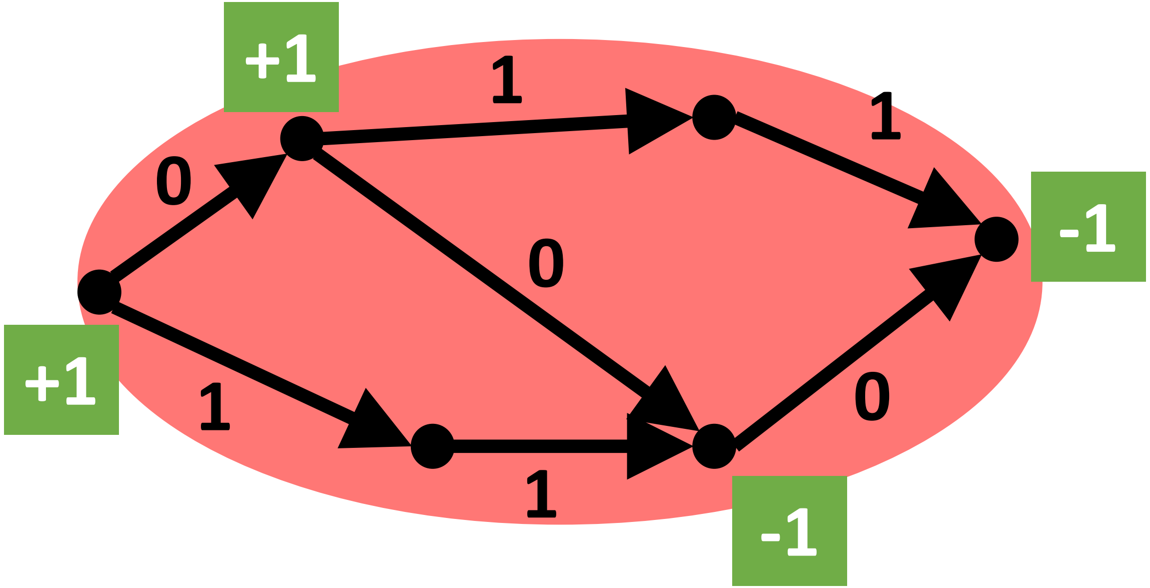

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$$\min \|x\|_1$$
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Minimum cost flow

Benchmark: Optimization on Graphs

$$\min |x|_{\infty}$$
$$Ax = b$$

max flow

$$\min |x|_1$$
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min cost flow

Benchmark: Optimization on Graphs

$$\min |x|_\infty$$
$$Ax = b$$

max flow

Q: ***Are these problems really that hard?***

$$\min |x|_1$$
$$Ax = b$$

min cost flow

Benchmark: Optimization on Graphs

min $|x|_\infty$
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Q: *Are these problems really that hard?*

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First order methods (gradient descent)

➜ running time strongly depends on matrix structure

➜ in general, takes time at least **$\Omega(m^{1.5}/\mathrm{poly}(\varepsilon))$**

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Second order methods (Newton method, IRLS)

➜ interior point method: $\tilde{O}(m^{1/2})$ linear system solves

➜ can be made $\tilde{O}(n^{1/2})$ with a lot of work **[Lee-Sidford ’14]**

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"Hybrid" method

➜ [Christiano-Kelner-Madry-Spielman-Teng '11] $\tilde{O}(m^{1/3}/\varepsilon^{11/3})$ linear system solves

➜ ~30 pages of description and proofs for complicated method

This work

Natural IRLS method runs in $\tilde{O}(m^{1/3}/\varepsilon^{2/3}+1/\varepsilon^2)$ iterations

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*** no matter what the structure of the underlying matrix is**

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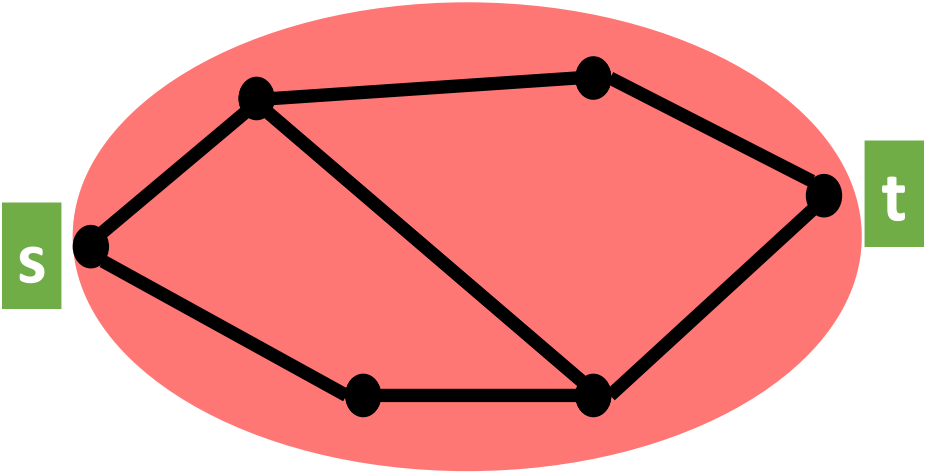

$$\min \|x\|_\infty \quad Ax = b \quad \leq OPT$$

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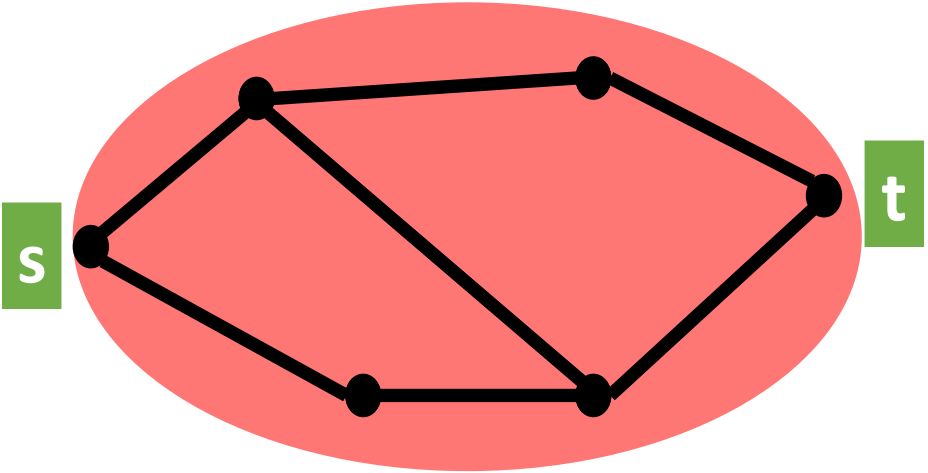

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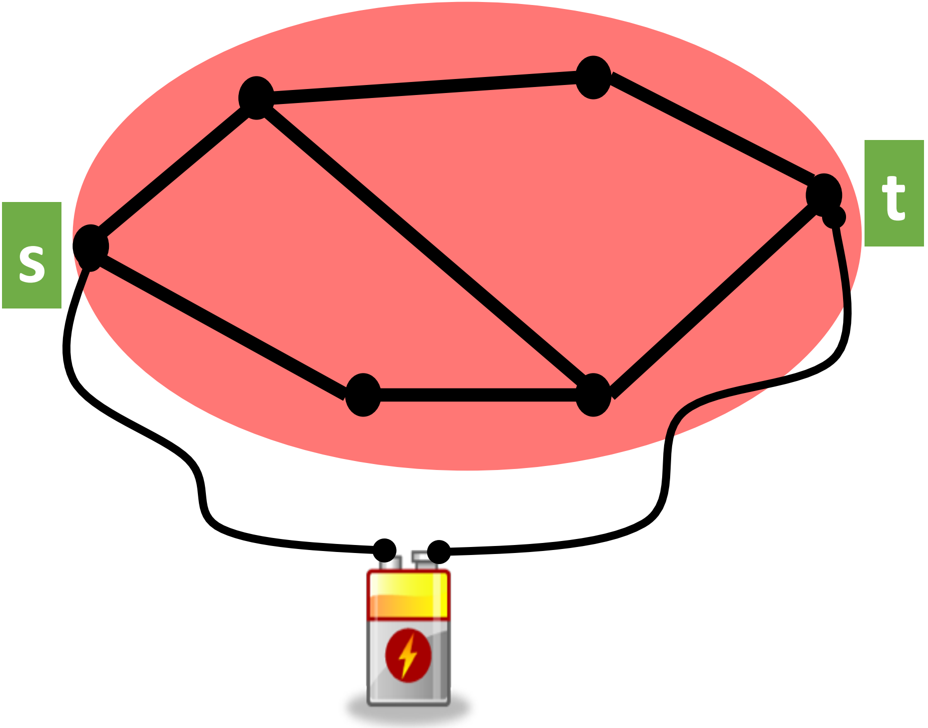

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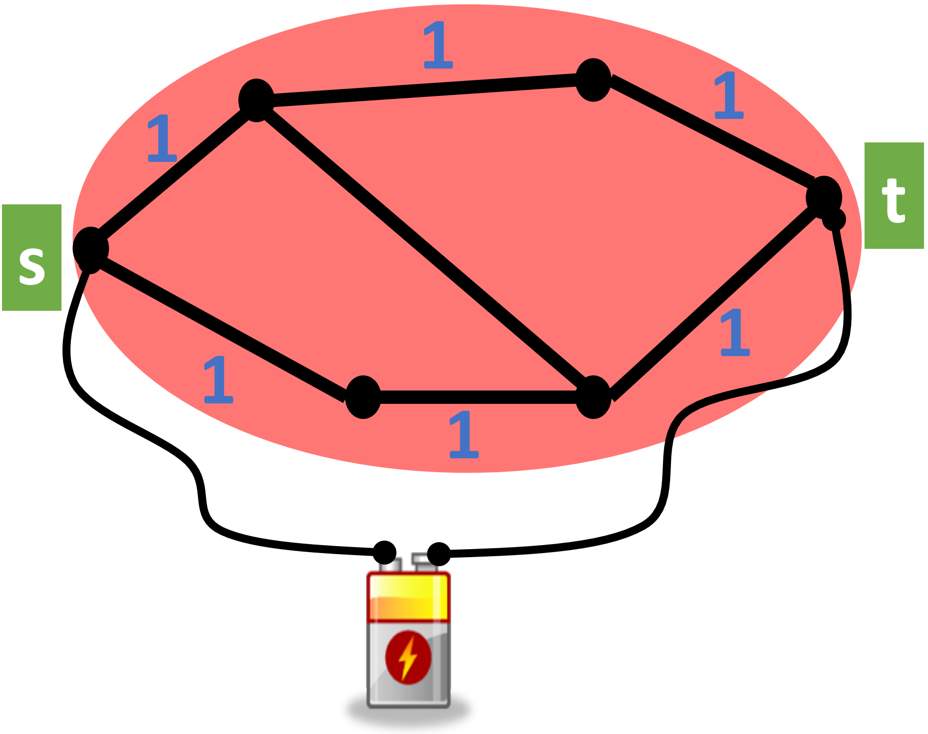

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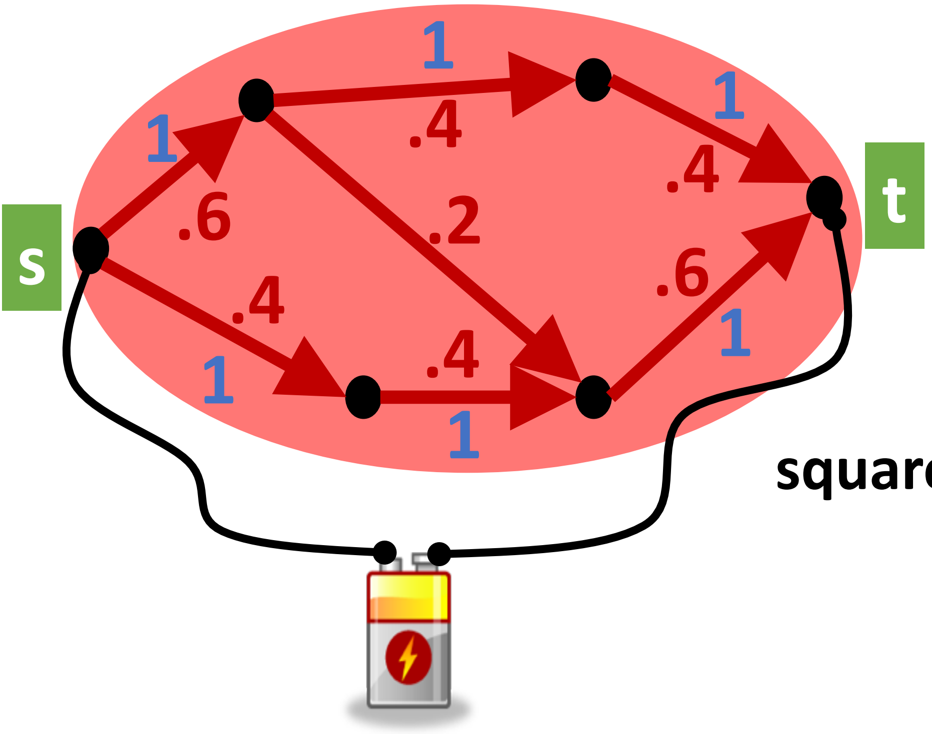

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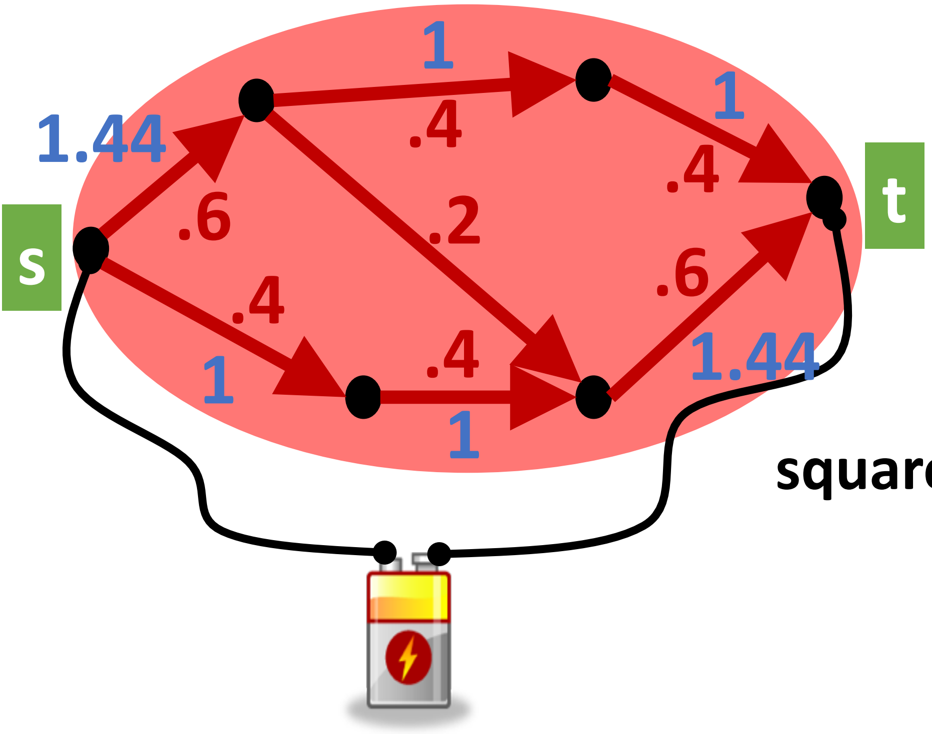

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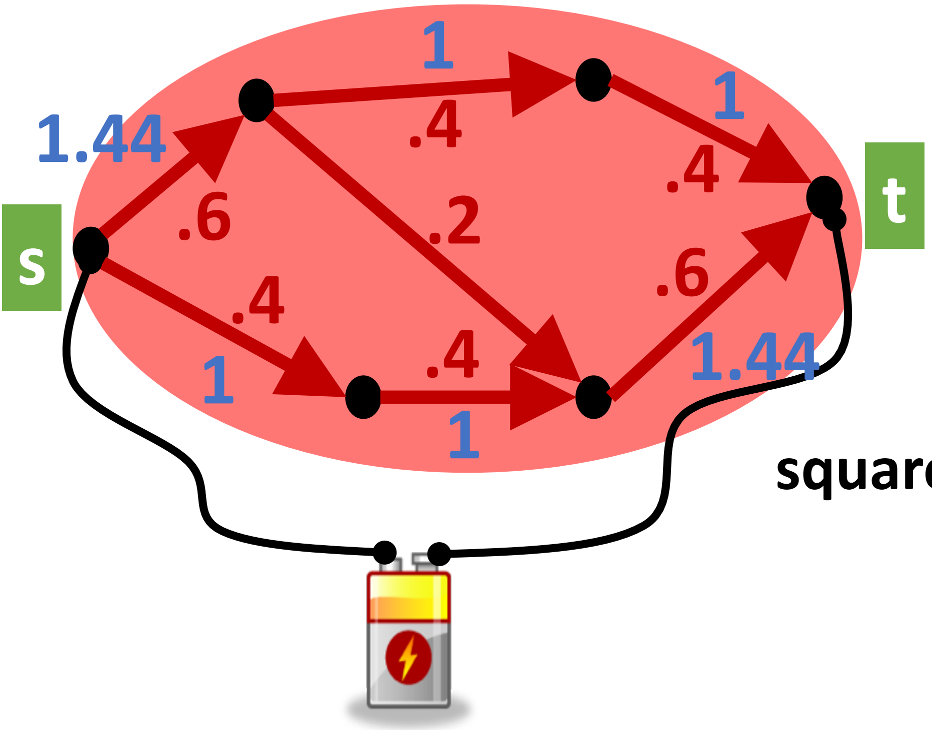

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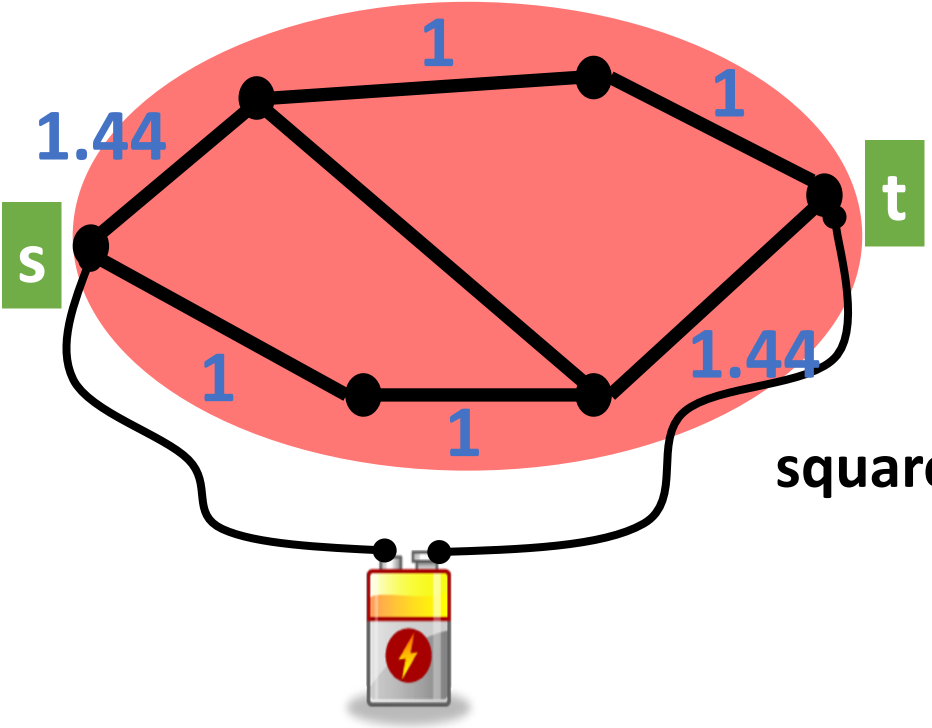

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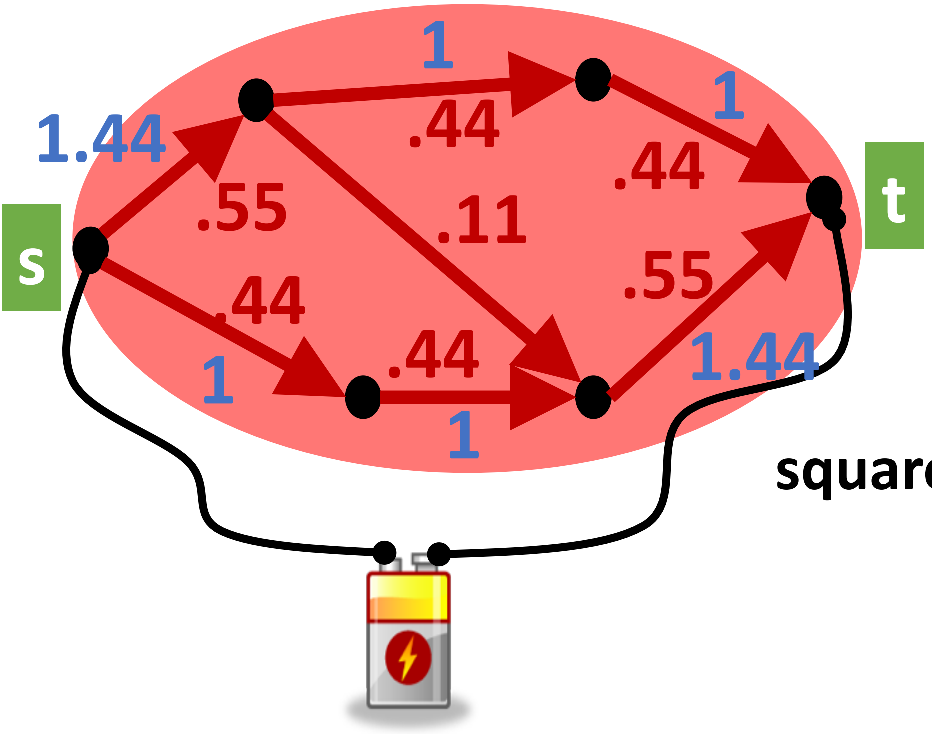

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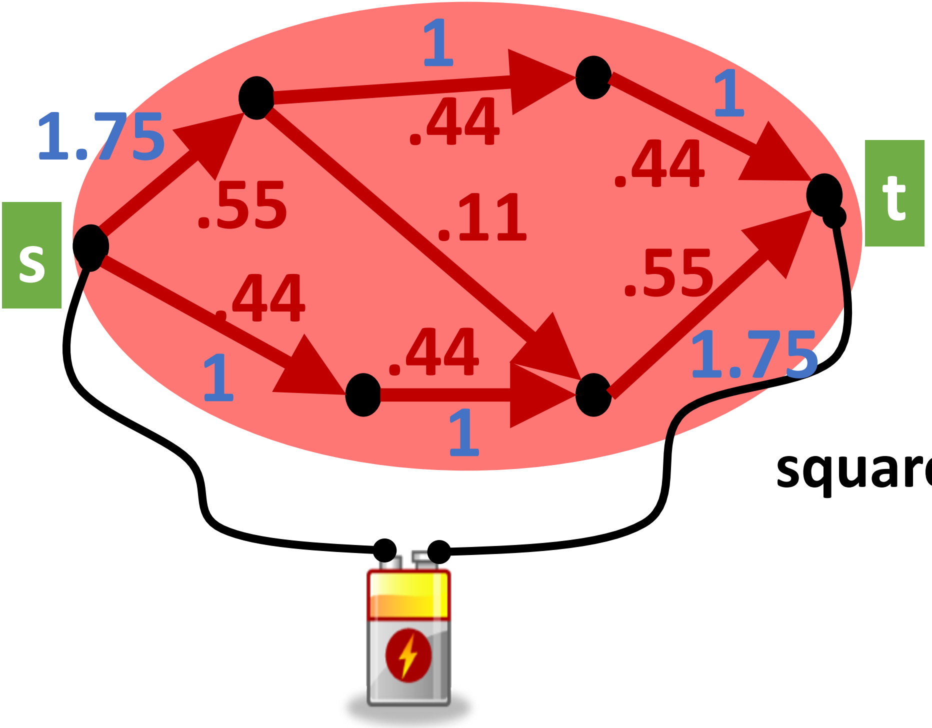

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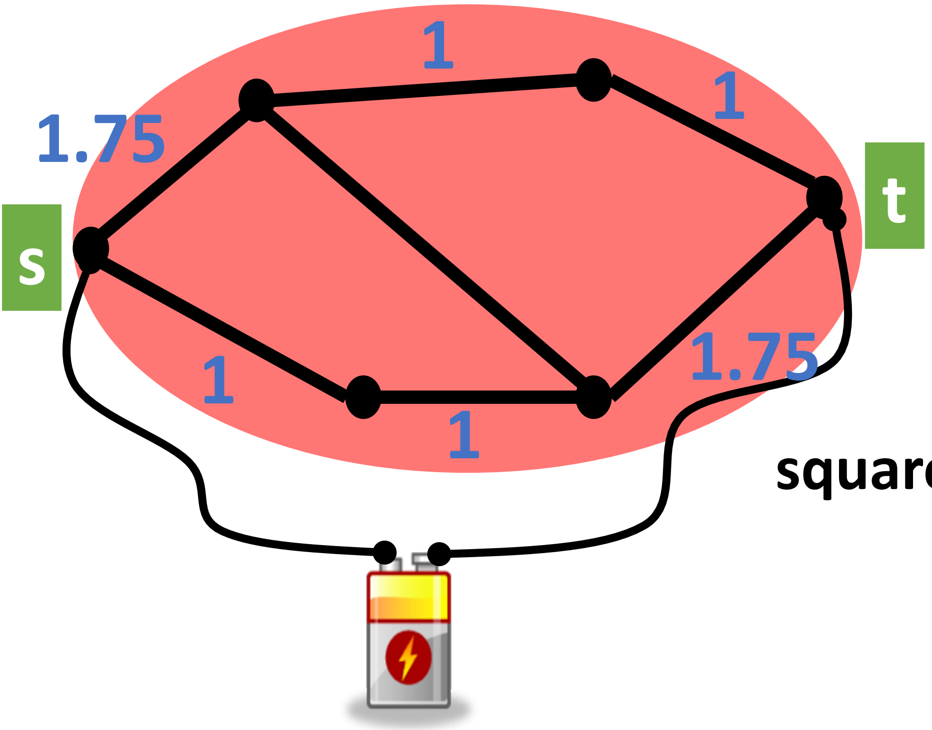

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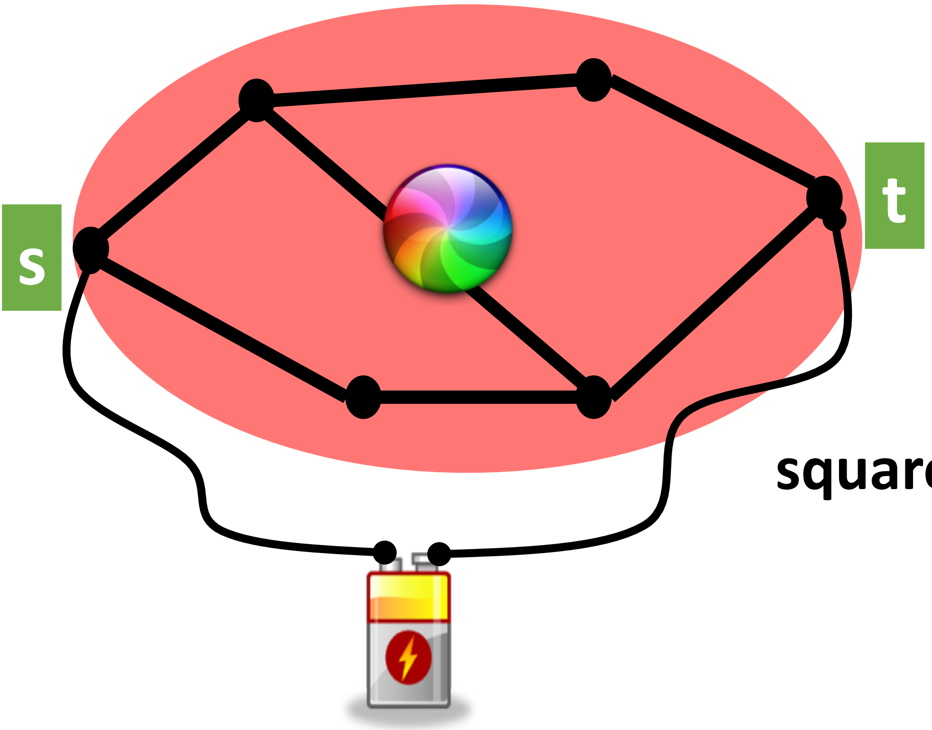

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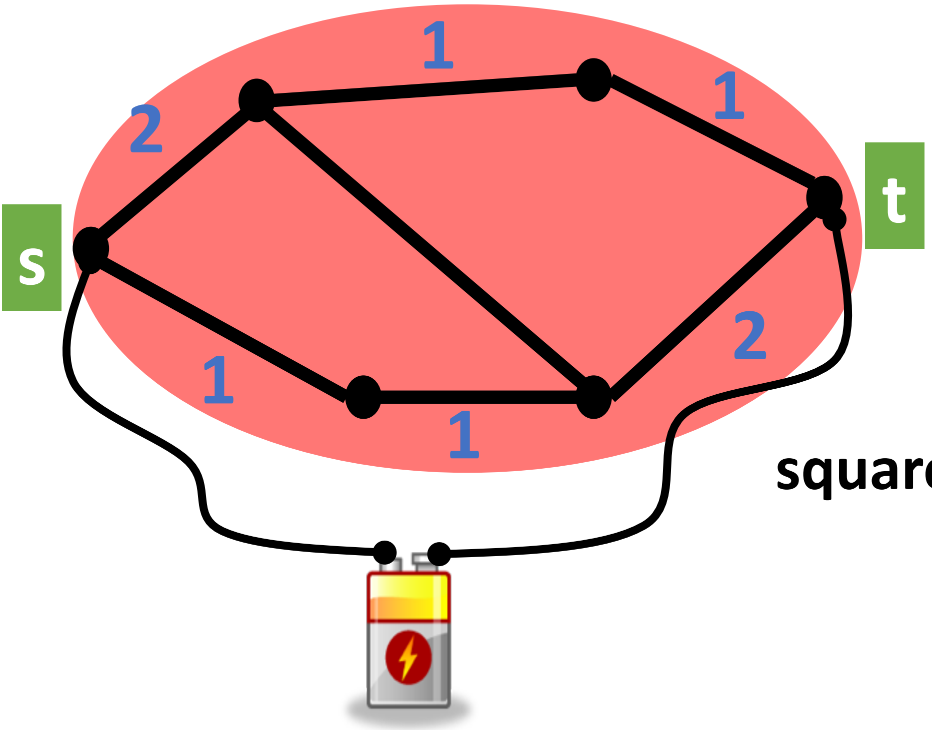

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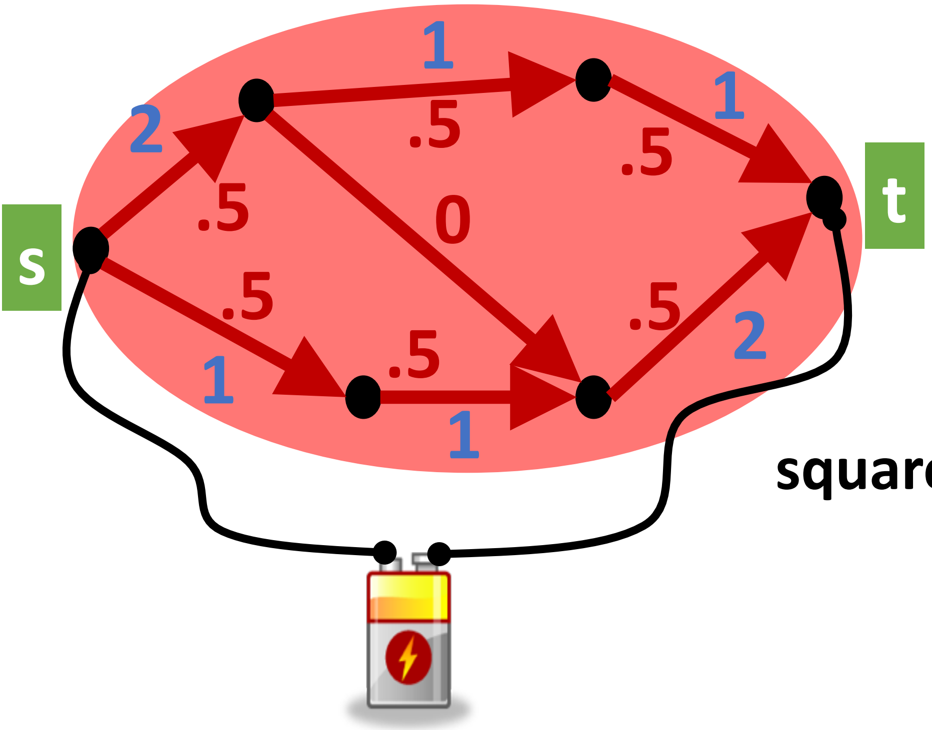

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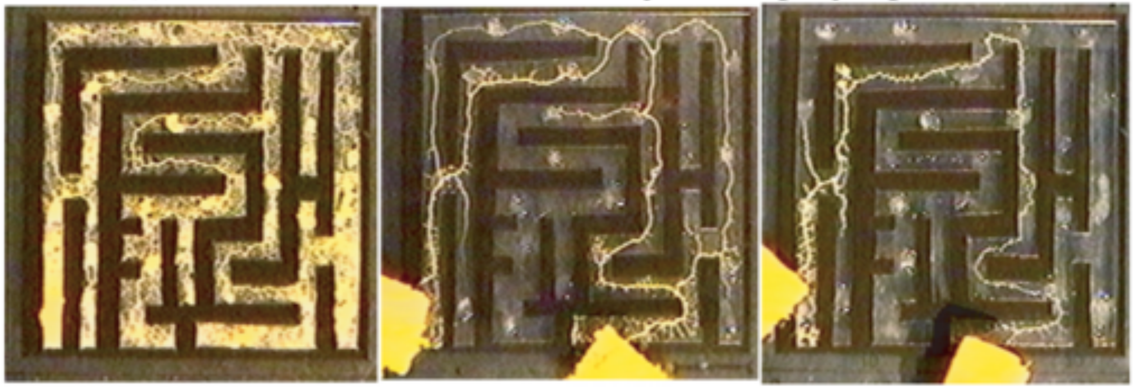

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- ➜ Any insights for new optimization methods?

Thank You!

More details at poster #208