Characterization of Convex Objective Functions and Optimal Expected Convergence Rates of SGD

Phuong Ha Nguyen[1]

Marten van Dijk[1], Lam M. Nguyen[2] and Dzung T. Phan[2]

Marten Lam P. Ha Dzung

1. Secure Computation Laboratory, ECE, University of Connecticut
2. IBM Research, Thomas J. Watson Research Center

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Problem Setting

- **Solve**

$$\min_{w \in R^d} \{F(w) = E_\xi[f(w;\xi)]\}$$

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  - **Smooth:**

$$||\nabla f(w;\xi) - \nabla f(w';\xi)|| \leq L||w - w'||$$

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Stochastic Gradient Descend (SGD):

Initialize: w_0
Iterate:
for $t = 0, 1, 2, \dots,$ **do**
  Choose $\eta_t > 0$
  Generate random ξ_t
  Compute $\nabla f(w_t;\xi_t)$
  Update $w_{t+1} = w_t - \eta_t \nabla f(w_t;\xi_t)$
end for

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- **Problem: Characterize Expected Convergence Rates**

$$E\left[\inf_{w_* \in W^*} ||w_t - w_*||^2\right] \text{ and } E[F(w_t) - F(w_*)]$$

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Beyond convex and strongly convex functions

Plain Convex

$F(w) - F(w_*) \geq 0$

?

Strongly Convex

$F(w) - F(w_*) \geq \frac{\mu}{2} ||w - w_*||^2$

ω-Convexity

Plain Convex

$$F(w) - F(w_*) \geq 0$$

ω − Convex

$$\omega\big(F(w) - F(w_*)\big) \geq \inf_{w_* \in W^*} ||w - w_*||^2,$$

$$\omega' > 0, \omega'' < 0,$$

Strongly Convex

$$F(w) - F(w_*) \geq \frac{\mu}{2}||w - w_*||^2$$

ω-Convexity with curvature $h \in [0,1]$

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$$\big(F(w) - F(w_*)\big)^h \geq \alpha \inf_{w_* \in W^*} ||w - w_*||^2$$

$h = 0$ ⟷ $h \in (0,1)$ ⟷ $h = 1$

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HEB (Holderian Error Bound): $\big(F(w) - F(w_*)\big)^h \geq \alpha \inf_{w_* \in W^*} ||w - w_*||^2$, where $h \in (0,2]$.

HEB and ω-convexity are not subclasses of one another but they do intersection for $h \in (0,1]$.
[Bolte, J., Nguyen, T. P., Peypouquet, J., and Suter, B. W. From error bounds to the complexity of first order descent methods for convex functions. Mathematical Programming, 165(2):471–507, Oct 2017]

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Close to optimal stepsize

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$h \in (0,1)$

SGD

$$\eta_t = \frac{c}{(t + \Delta)^{1/(2-h)}}$$

Close to optimal stepsize

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Convergence Rate of SGD

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$$\eta_t = \frac{c}{(t+\Delta)^{1/(2-h)}}$$

Close to optimal stepsize

$$E\left[\inf_{w_* \in W^*} ||w_t - w_*||^2\right] = O\big(t^{-h/(2-h)}\big)$$

$$\frac{1}{t}\sum_{i=t+1}^{2t} E[F(w_i) - F(w_*)] = O(t^{-1/(2-h)})$$

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$$E\left[\inf_{w_* \in W^*} ||w_t - w_*||^2\right] = O\big(t^{-h/(2-h)}\big)$$ [Useless,0] ⟷ [Useful,1]

$0 \leftarrow \quad h \quad \rightarrow 1$

$$\frac{1}{t}\sum_{i=t+1}^{2t} E[F(w_i) - F(w_*)] = O(t^{-1/(2-h)})$$ [Useful,0] ⟷ [Useful,1]

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$$E\left[\inf_{w_* \in W^*} ||w_t - w_*||^2\right] = O\big(t^{-h/(2-h)}\big)$$

$$\frac{1}{t}\sum_{i=t+1}^{2t} E[F(w_i) - F(w_*)] = O(t^{-1/(2-h)})$$

h= ½

$$F(w) = H(w) + \lambda G(w), H(w) - convex$$

$$G(w) = \sum_{i=1}^{d} [e^{w_i} + e^{-w_i} - 2 - w_i^2]$$

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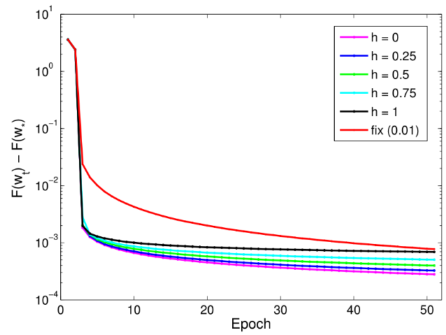
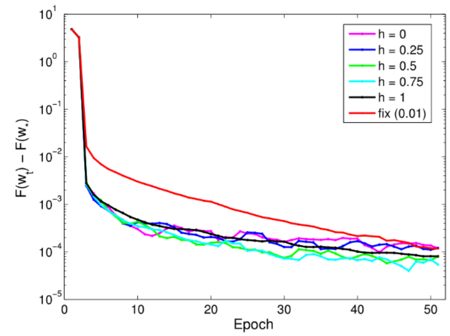
Experiment

Curvature 0 (convex)

$$f_i(w) = \log(1 + \exp(-y_i x_i^T w))$$



(a)



(b)

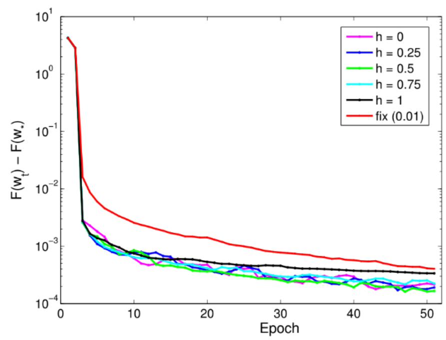
Curvature unknown

$$f_i^a(w) = f_i(w) + \lambda||w||$$

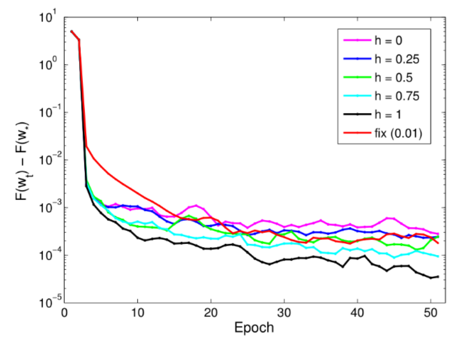
Curvature ½

$$f_i^a(w) = f_i(w) + \lambda G(w)$$

$$G(w) = \sum_{i=1}^{d} [e^{w_i} + e^{-w_i} - 2 \ - w_i^2]$$



(c)



(d)

Curvature 1 (strongly convex)

$$f_i^c(w) = f_i(w) + \frac{\lambda}{2}||w||^2$$

Conclusion

- ω-convexity notion: plain convex, strongly convex and something in between
- SGD with ω-convex objective functions

Thank you for your attention! ☺

https://arxiv.org/abs/1810.04100

Poster Number: #193 – Pacific Ballroom. – 06:30—09:00PM – 06/11