Monge blunts Bayes:
Hardness Results for Adversarial Training

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Overview

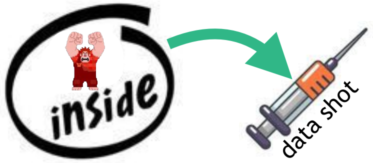
- Hardness results on adversarial training. ***Key result*** applicable to a learner:
  - optimising ***any*** loss satisfying a mild statistical requirement, and
  - learning a classifier from ***any*** class satisfying a mild continuity assumption

- Implementation disentangles adversarial training:
  1. generate adversarial data (***Key result*** solves the compression of an OT plan)

     —//—

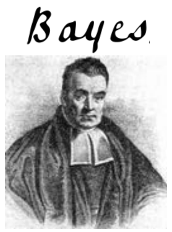
  2. training as usual

- Toy experiments against “weakly activated” adversarial data reveal generalisation improves *on clean data as well*



Key players: Bayes

Bayes



1- Classifiers

$$\mathcal{H} \subseteq \mathbb{R}^{\mathcal{X}}$$

← domain

← sign = class

2- Adversaries

$$\mathcal{A} \subseteq \mathcal{X}^{\mathcal{X}}$$

← domain

3- (differentiable) Loss

$$\ell : \{-1, 1\} \times [0, 1] \to \overline{\mathbb{R}}$$

(conditional) Bayes risk

$$\underline{L}(\pi) \doteq \inf_c \mathsf{E}_{\mathsf{Y}\sim\pi} \ell(\mathsf{Y}, c) \Rightarrow$$ ***proper***

(π in inf)

composite loss

link $\psi : [0, 1] \to \mathbb{R}$

$$\ell_\psi(y, v) \doteq \ell(y, \psi^{-1}(v))$$

canonical loss

$$\psi \doteq -\underline{L}'$$

(ψ “hidden”)

blunt predictor

$$h^\circ \doteq \psi\left(\tfrac{1}{2}\right) \quad = 0$$ (often)

corresponding loss: $\ell^\circ_\psi = \ell^\circ$

4- general adversarial loss

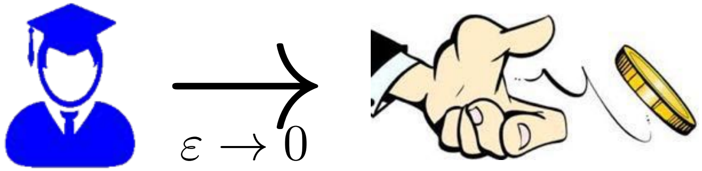
$$\ell(\mathcal{H}, \mathcal{A}, D) \doteq \min_{h \in \mathcal{H}} \mathsf{E}_{(\mathsf{X},\mathsf{Y})\sim D}\left[\max_{a \in \mathcal{A}} \ell(\mathsf{Y}, h \circ a(\mathsf{X}))\right]$$

particular case, Madry *et al.*’18:

$$\boldsymbol{x} \overset{a}{\to} \boldsymbol{x} + \boldsymbol{\delta} \text{ s.t. } \|\boldsymbol{\delta}\| \leq \delta^*$$

$\mathcal{H}$ **ε-defeated by $\mathcal{A}$ on ℓ** iff

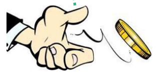
$$\ell(\mathcal{H}, \mathcal{A}, D) \geq (1 - \varepsilon) \cdot \ell^\circ$$



Main negative result

- For any proper composite loss ℓ, classifiers $\mathcal{H}$, adversaries $\mathcal{A}$ (+integrability assumptions),

$$\ell(\mathcal{H}, \mathcal{A}, D) \geq \left(\ell^{\circ} - \frac{1}{2} \cdot \min_{a \in \mathcal{A}} \beta_a \right)_+$$



$$\beta_a \doteq \max_{h \in \mathcal{H}} \{ \varphi(\underset{\text{“+” ex.}}{P}, f, \pi, 2\underline{L}(1)) - \varphi(\underset{\text{“-” ex.}}{N}, f, 1-\pi, -2\underline{L}(0)) \} \text{ with } \varphi(Q, f, u, v) \doteq \int_{\mathcal{X}} u \cdot (f(\boldsymbol{x}) + v) \mathrm{d}Q(\boldsymbol{x})$$

and $f \doteq (-\underline{L}') \circ \psi^{-1} \circ h \circ a$

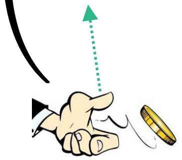
Example: if $\underline{L}(0) = \underline{L}(1)$ and $\pi = 1/2$, then β_a is $\propto$ Integral Probability Metric for class $\{(-\underline{L}') \circ \psi^{-1} \circ h \circ a : h \in \mathcal{H}\}$

(more in paper / poster)

Main negative result — consequence #1

- For any proper composite loss ℓ, classifiers $\mathcal{H}$, adversaries $\mathcal{A}$ (+integrability assumptions),

$$\ell(\mathcal{H}, \mathcal{A}, D) \geq \left(\ell^\circ - \frac{1}{2} \cdot \min_{a \in \mathcal{A}} \beta_a\right)_+$$



$\beta_a \doteq \max_{h\in\mathcal{H}}\{$ … $u \cdot (f(\boldsymbol{x}) + v)\mathrm{d}Q(\boldsymbol{x})$

… $h \circ a$

Example: if $\underline{L}(0)$ … for class $\{(-\underline{L}') \circ \psi^{-1} \circ h \circ a : h \in \mathcal{H}\}$

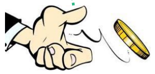
Hence, if $\exists a \in \mathcal{A}$ such that $\beta_a \leq 2\varepsilon\ell^\circ$ then $\mathcal{H}$ is ε-defeated by $\mathcal{A}$ on ℓ

Main negative result — consequence #2

- For any proper composite loss ℓ, classifiers $\mathcal{H}$, adversaries $\mathcal{A}$ (+integrability assumptions),

$$\ell(\mathcal{H}, \mathcal{A}, D) \geq \left(\ell^{\circ} - \frac{1}{2} \cdot \min_{a \in \mathcal{A}} \beta_a\right)_+$$

$$= \min_{h \in \mathcal{H}} \mathsf{E}_{(\mathsf{X},\mathsf{Y})\sim D} \left[\max_{a \in \mathcal{A}} \ell(\mathsf{Y}, h \circ a(\mathsf{X}))\right]$$



$\beta_a \doteq \max_{h \in \mathcal{H}} \{$ … $u \cdot (f(\boldsymbol{x}) + v)\mathrm{d}Q(\boldsymbol{x})$

$h \circ a$

Example: if $\underline{L}(0)$ … ass $\{(-\underline{L}') \circ \psi^{-1} \circ h \circ a : h \in \mathcal{H}\}$

RHS: roles permuted — adversary in *outer* optimisation suggests 2-stages optimisation to train classifier:

(i) (build adversary) craft *adversarial training data*

(ii) **train from adversarial data**

Adversaries 1/3: MMD

- Direct link with Maximum Mean Discrepancy (MMD)
  - Let $\mathcal{H}$ be the unit ball of a RKHS w/ reproducing kernel κ.

Adversarial mean embedding of a on Q

$$\mu_{a,Q} \doteq \int_{\mathcal{X}} \kappa(a(\boldsymbol{x}), .)\mathrm{d}Q(\boldsymbol{x})$$

(*Adversarial*) MMD between P and N

$$\mathrm{MMD}[P, N|a] \doteq \|\mu_{a,P} - \mu_{a,N}\|_{\mathcal{H}}$$

(if $\underline{L}(0) = \underline{L}(1)$ and $\pi = 1/2$)

$$\beta_a = \frac{1}{4} \cdot \mathrm{MMD}[P, N|a]$$

$\mathcal{H}$ is ε-defeated by $\mathcal{A}$ on ℓ if

$$\exists a \in \mathcal{A} \text{ s.t. } \mathrm{MMD}[P, N|a] \leq 8\varepsilon\ell^{\circ}$$

(more in paper / poster)

Adversaries 2/3: Monge

vs

- Allows to build efficient adversaries when classifiers are Lipschitz
  - *Solve the compression of an optimal transport plan*

(*Adversarial*) OT plan between P and N

$$C(a, P, N) \doteq \inf_{\mu \in \Pi(P,N)} \int c(a(\boldsymbol{x}), a(\boldsymbol{x}'))\mathrm{d}\mu(\boldsymbol{x}, \boldsymbol{x}')$$

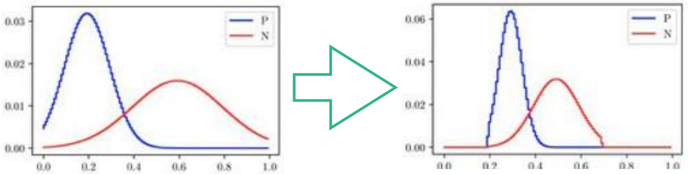
Monge efficiency (for cost $c : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$)

$\mathcal{A}$ **δ-Monge efficient for c on P, N iff**

$$\exists a \in \mathcal{A} : C(a, P, N) \leq \delta$$

Suppose $\mathcal{H}$ is K-Lipschitz with respect to c, $\mathcal{A}$ is δ-Monge efficient for c on P, N. Suppose $\underline{L}(0) = \underline{L}(1)$, $\pi = 1/2$.

If $\delta \leq 8\varepsilon\ell^\circ / K$, then $\mathcal{H}$ is $\mathcal{E}$-defeated by $\mathcal{A}$ on ℓ



(more general statement in paper / poster)

Adversaries 3/3: Boosting

A. It is possible to $\mathcal{E}$-defeat $\mathcal{H}$ *simultaneously* on a whole set $\mathcal{L}$ of **symmetric** losses

Simple way to defeat strategies learning/tuning the loss
— important case because common losses fit in (log, square, Matsushita, etc.)

B. It is possible to craft **very strong** adversaries from **very weak** ones

RKHS example — suppose there exists a weakly contractive adversary a in a feature map Φ of the RKHS: $\|\Phi \circ a(\boldsymbol{x}) - \Phi \circ a(\boldsymbol{x}')\|_{\mathcal{H}} \leq (1-\eta) \cdot \|\Phi(\boldsymbol{x}) - \Phi(\boldsymbol{x}')\|_{\mathcal{H}}, \forall \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X}$
Then $\forall \delta > 0$**, composing just** $(1/\eta) \cdot \log(W_1^{\Phi}/\delta)$ **adversaries yields** δ**-Monge efficiency**

W_1^{Φ} = 1-Wasserstein distance between P and N in Φ

(complete statements in paper / poster)

Take home theoretical messages

vs

A. Replace adversarial training by **training from adversarial data**

B. If loss in specific classes, incl. popular losses, adversary can be **loss agnostic**

C. If learner's $\mathcal{H}$ is Lipschitz, use Lipschitz cost in an **OT compression problem**

D. **Adversarial boosting:** craft strong adversaries from weak adversaries

Some Monge efficient adversaries

A. *Mixup* adversaries (named after Zhang, Cissé, Dauphin & Lopez-Paz ’18)

general transformation: $a(\boldsymbol{x}) \doteq (1-\lambda)\cdot\boldsymbol{x} + \lambda\cdot\boldsymbol{x}'$

neighbour in other class

cluster / class centroid

sample centroid, etc.

B. Monge adversary – for a tight control on Monge efficiency, focus on

$$\min_{a} \text{Wasserstein-OT s.t. } d(a(\boldsymbol{x}),\boldsymbol{x}) \leq \underset{\text{“budget”}}{\alpha}, \forall\boldsymbol{x}$$

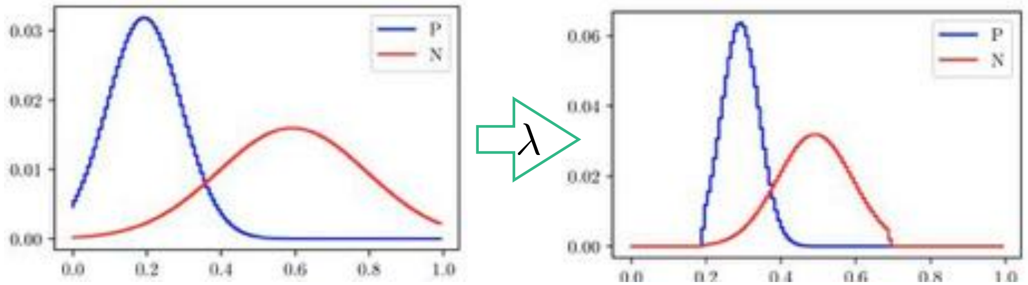
(more in paper / poster)

Toy experiments 1/2 — data & transformations

A. *Mixup*-to-sample-centroid

$$x' = \mathrm{E}_D[\mathrm{X}]$$

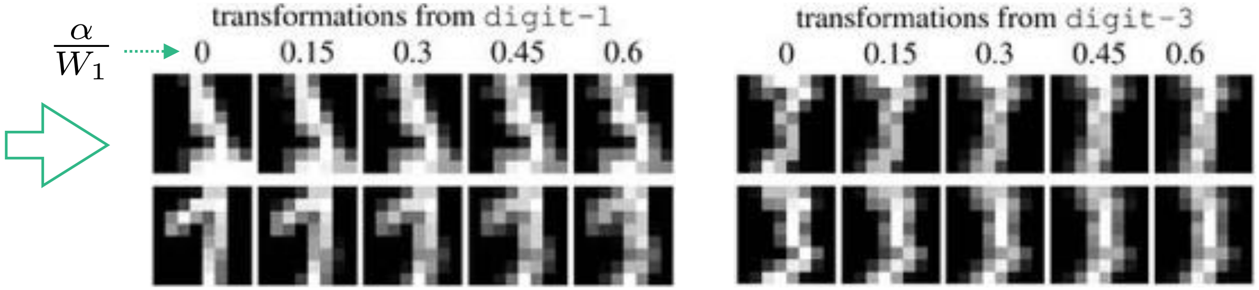
1D normal classes



B. Monge adversary for Wasserstein = W_2^2 and $d = \|.\|_1$ (cvx)

USPS data

classes 1 & 3


transformations from `digit-1`

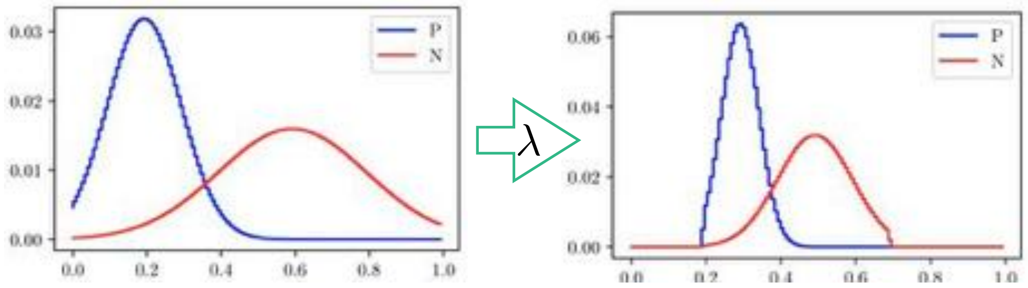
transformations from `digit-3`

Toy experiments 2/2 — findings

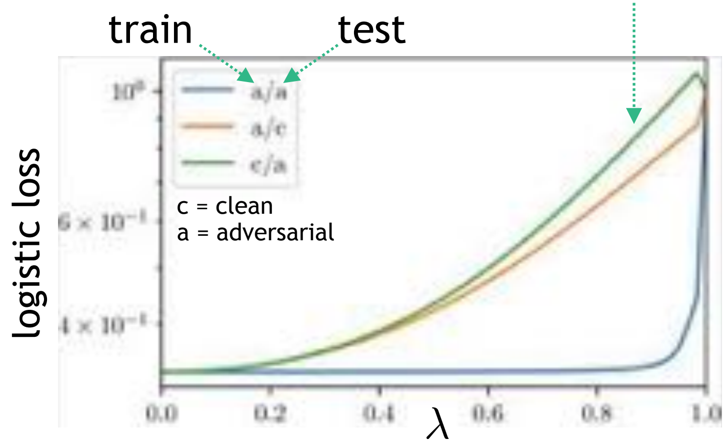
A. *Mixup*-to-sample-centroid

$$x' = \mathrm{E}_D[\mathrm{X}]$$

1D normal classes



worst trains on *c* !

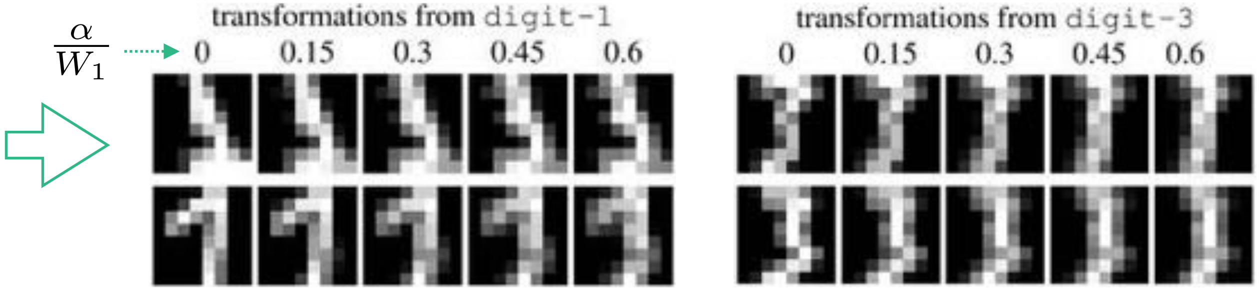


c = clean
a = adversarial

B. Monge adversary for Wasserstein = W_2^2 and $d = \|.\|_1$ (cvx)

USPS data

classes 1 & 3



best trains on (weak) *a*
***(even tested on c)* !**

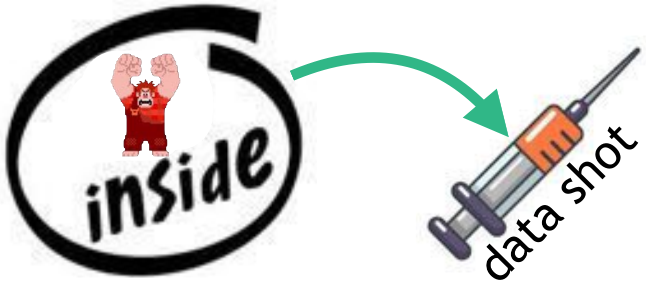
α/d	c/c	c/a	a/c	a/a
0.15	0.03	0.11	**0.00**	**0.02**
0.30	0.03	0.25	**0.00**	0.12
0.45	0.03	0.48	**0.01**	0.55
0.60	0.03	0.74	0.20	0.96

Conclusion

- Replacement of adversarial training by training from adversarial data
- Adversaries that can be effective against wide ranges of $(\mathcal{H}, \ell)$

- Adversarial strategy against Lipschitz classifiers: **compression of OT plans** (between class marginals)

- **Toy experiments** reveal that *sufficiently weak adversarial data* can **improve generalisation on clean data**



- Next step: explain such a “vaccination phenomenon”

Thank you

(get your data shot at poster # 191)