

Flat Metric Minimization with Applications in Generative Modeling

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Technical
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Motivation

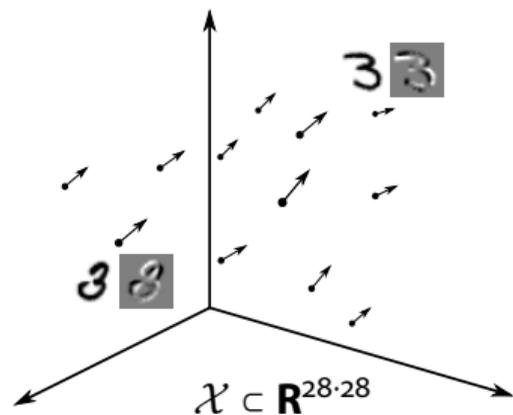
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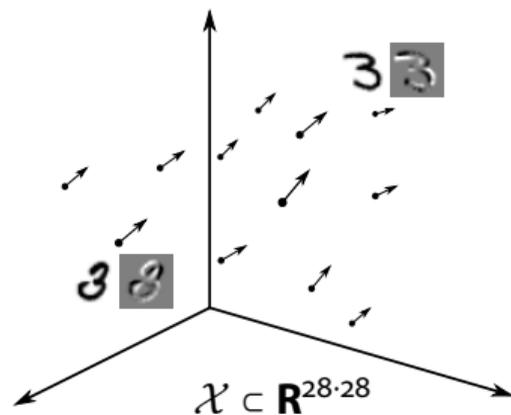


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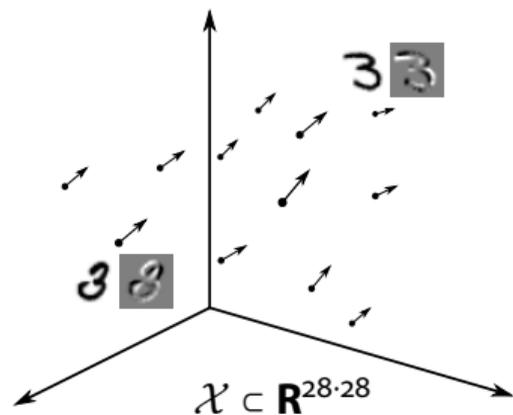


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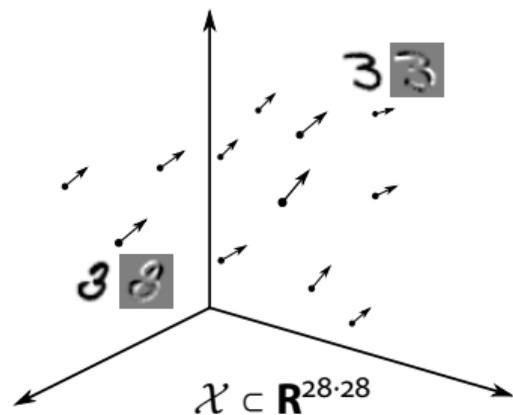


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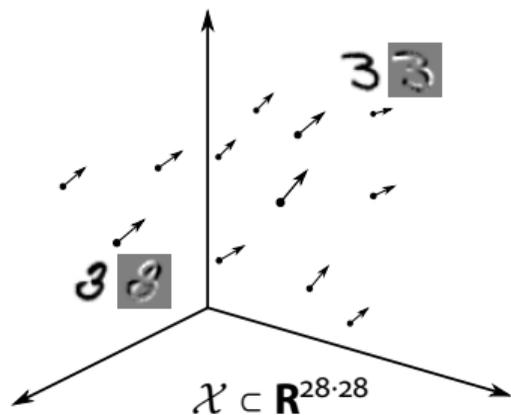
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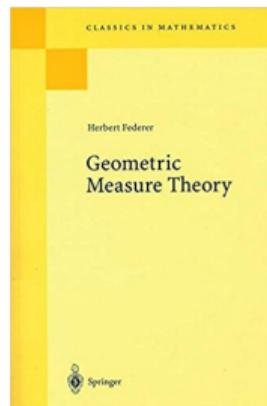
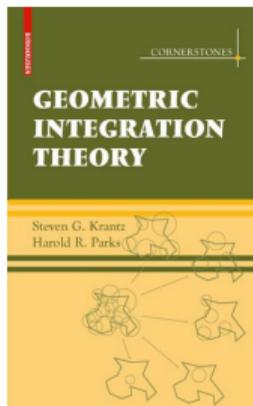
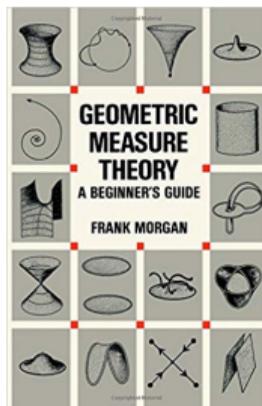
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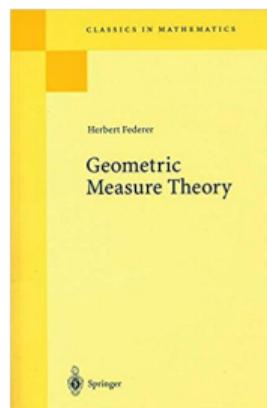
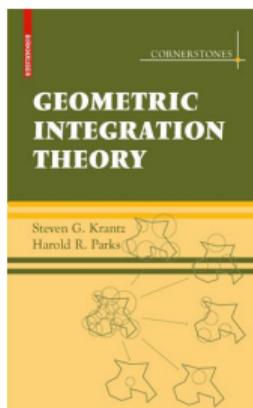
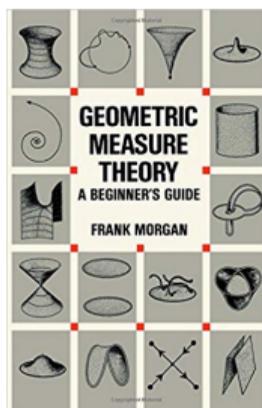
Contributions:

- ▶ We propose the novel perspective to represent oriented data with *k-currents* from *geometric measure theory*.
- ▶ Using this viewpoint within the context of GANs, we learn a generative model which behaves *equivariantly* to specified tangent vectors.

An invitation to geometric measure theory (GMT)

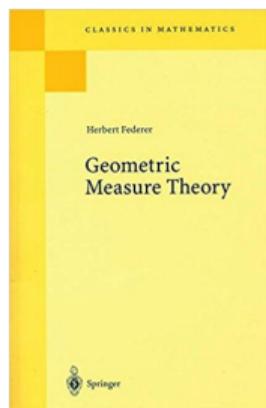
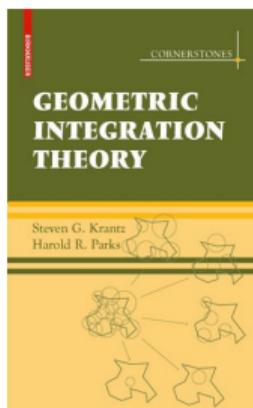
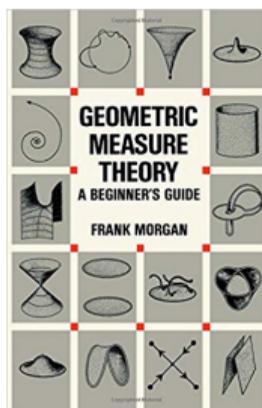


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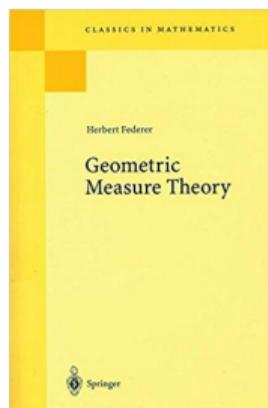
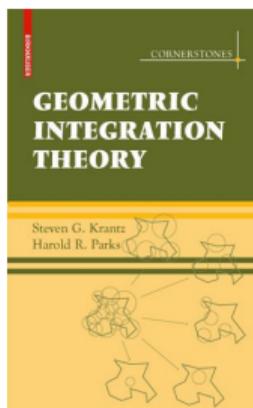
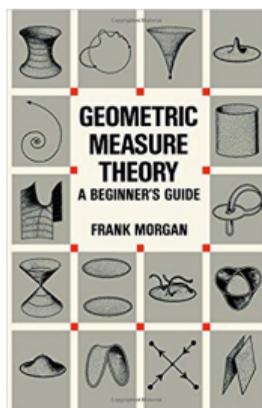
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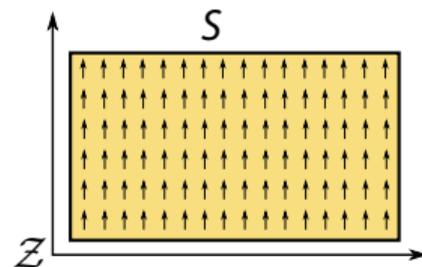
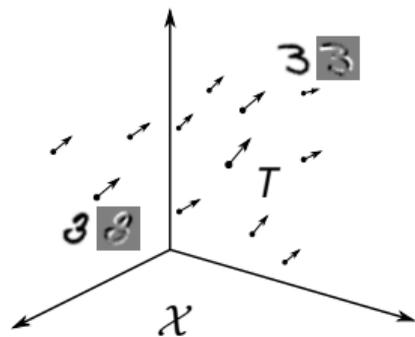
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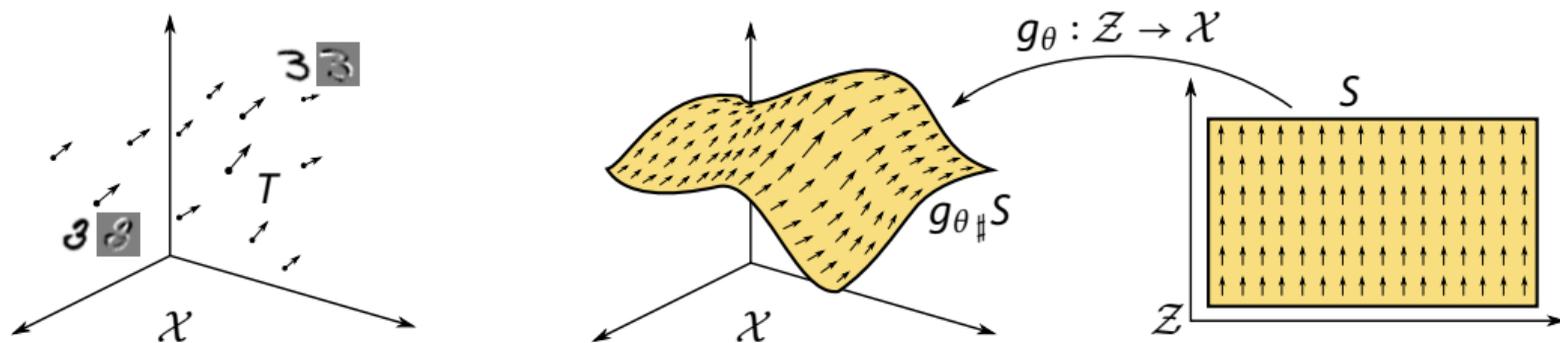
- ▶ Differential geometry, generalized through measure theory to deal with surfaces that are not necessarily smooth.
- ▶ k -currents \approx generalized (possibly quite irregular) oriented k -dimensional surfaces in d -dimensional space.
- ▶ The class of currents we consider form a *linear space*. It includes oriented k -dimensional surfaces as elements.

Generalizing Wasserstein GANs to k -currents



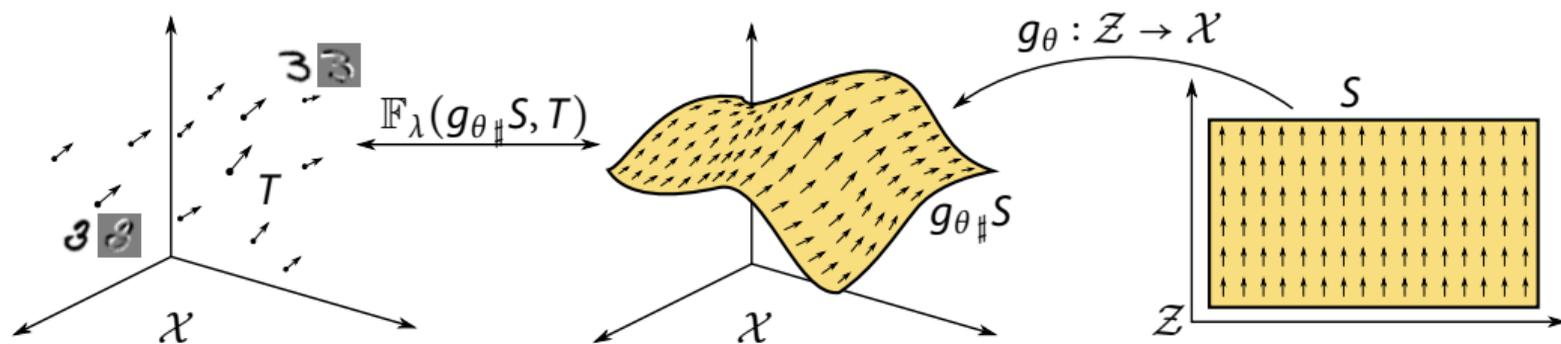
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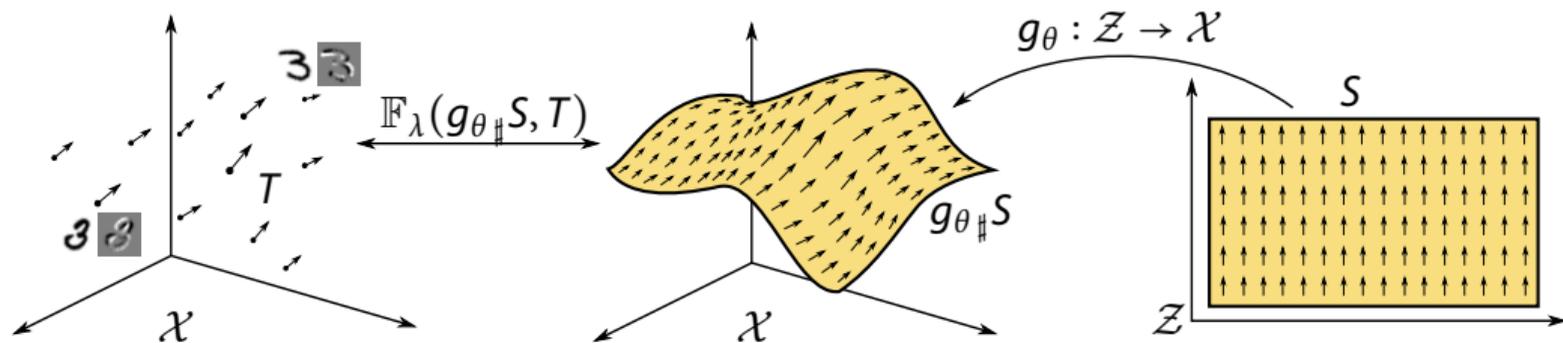
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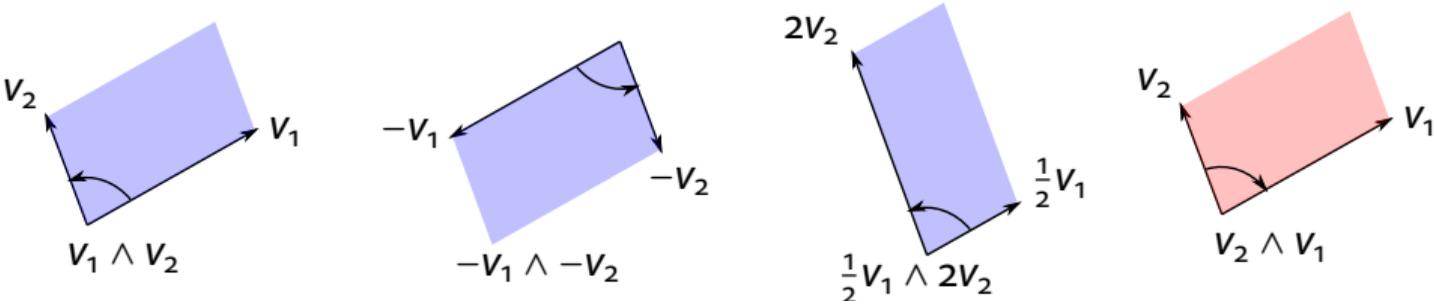
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- ▶ For $k = 0$ the flat metric is closely related to the Wasserstein-1 distance and positive 0-currents with unit mass are probability distributions.

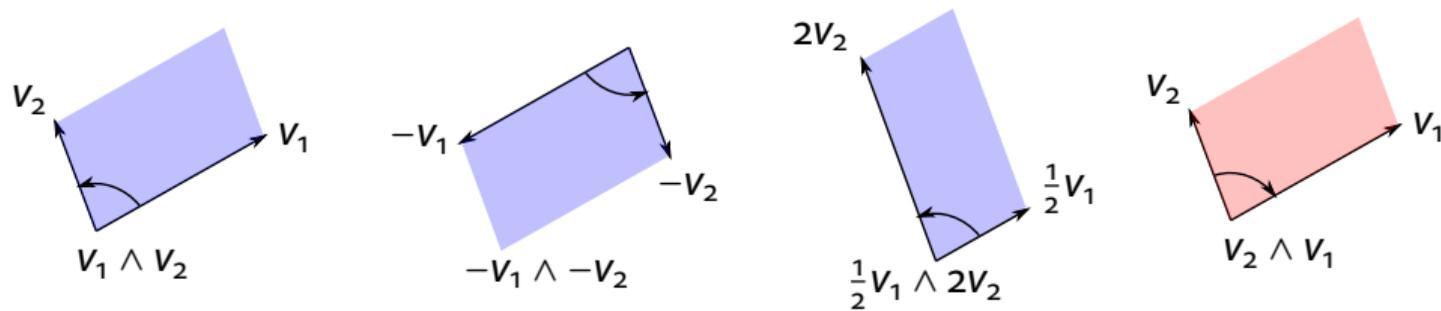
k -dimensional orientation in d -dimensional space

- Simple k -vectors $v = v_1 \wedge \dots \wedge v_k \in \Lambda_k \mathbf{R}^d$ describe oriented k -dimensional subspaces together with an area in \mathbf{R}^d :



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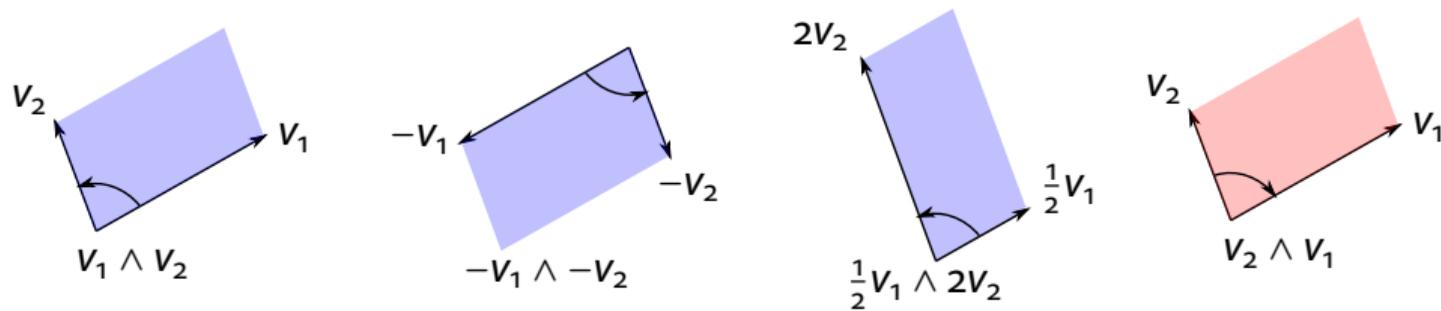
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- The set of simple k -vectors forms a nonconvex cone in the vector space $\Lambda_k \mathbf{R}^d$.
- For $v = v_1 \wedge \cdots \wedge v_k$, $w = w_1 \wedge \cdots \wedge w_k$:

$$\langle v, w \rangle = \det(V^T W), \quad |v| = \sqrt{\langle v, v \rangle}.$$

Oriented manifolds, differential forms and currents

- ▶ *Orientation* of a k -dimensional manifold \mathcal{M} : continuous simple k -vector map $\tau_{\mathcal{M}} : \mathcal{M} \rightarrow \Lambda_k \mathbf{R}^d$, $|\tau_{\mathcal{M}}(z)| = 1$ and $T_z \mathcal{M}$ "spanned" by $\tau_{\mathcal{M}}(z)$

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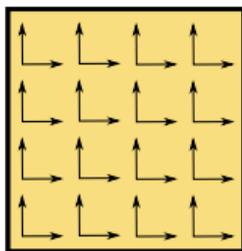
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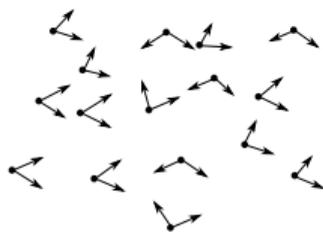
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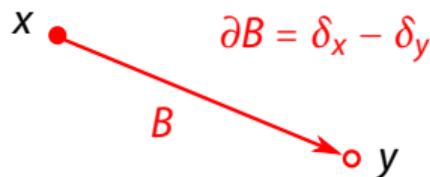
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- ▶ A geometric view on the Wasserstein-1 distance:

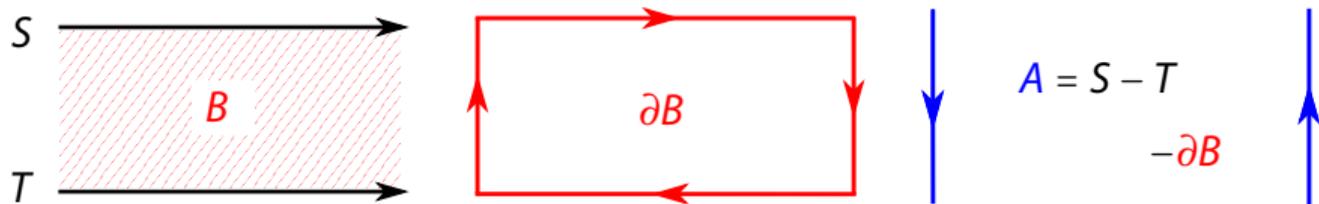
$$\mathcal{W}_1(S, T) = \min_{\partial B = S - T} \mathbb{M}(B). \text{ Example: } S = \delta_x, T = \delta_y:$$



The flat metric

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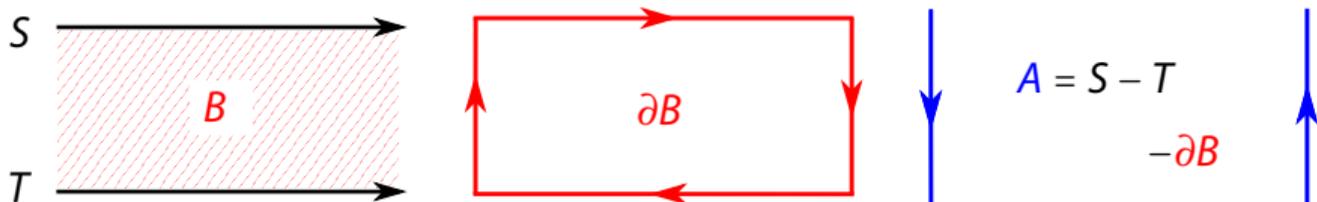
$$\mathbb{F}_\lambda(S, T) = \min_{S-T=\partial B+A} \mathbb{M}(B) + \lambda \mathbb{M}(A) = \sup_{\substack{\|\omega\|^* \leq \lambda \\ \|d\omega\|^* \leq 1}} (S - T)(\omega).$$



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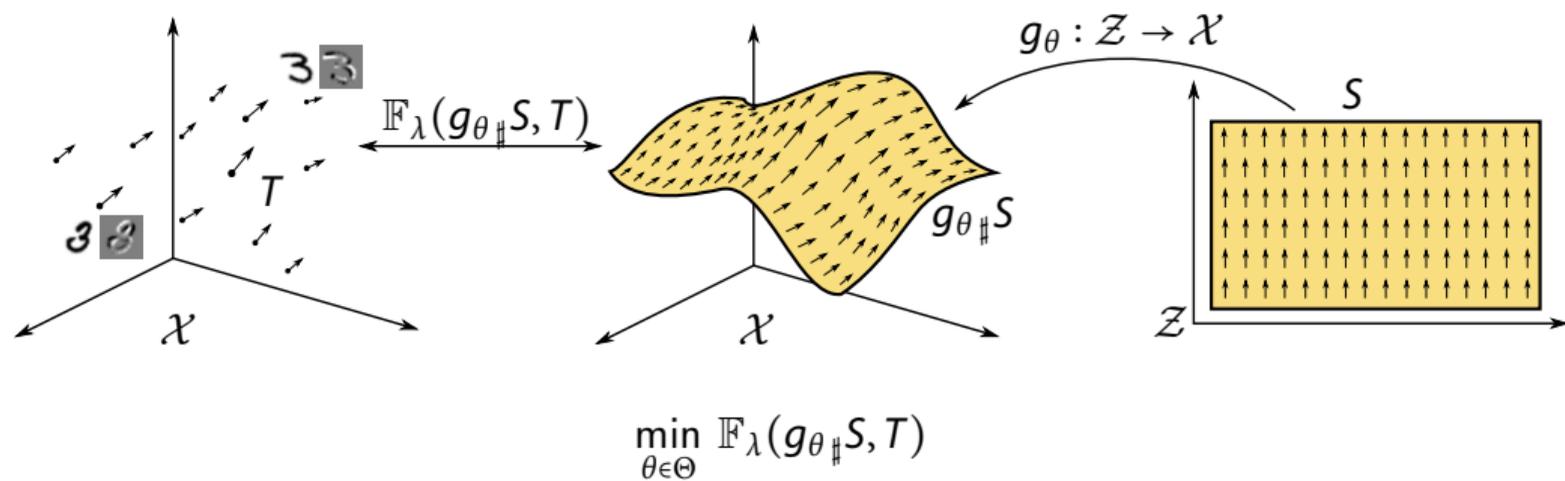
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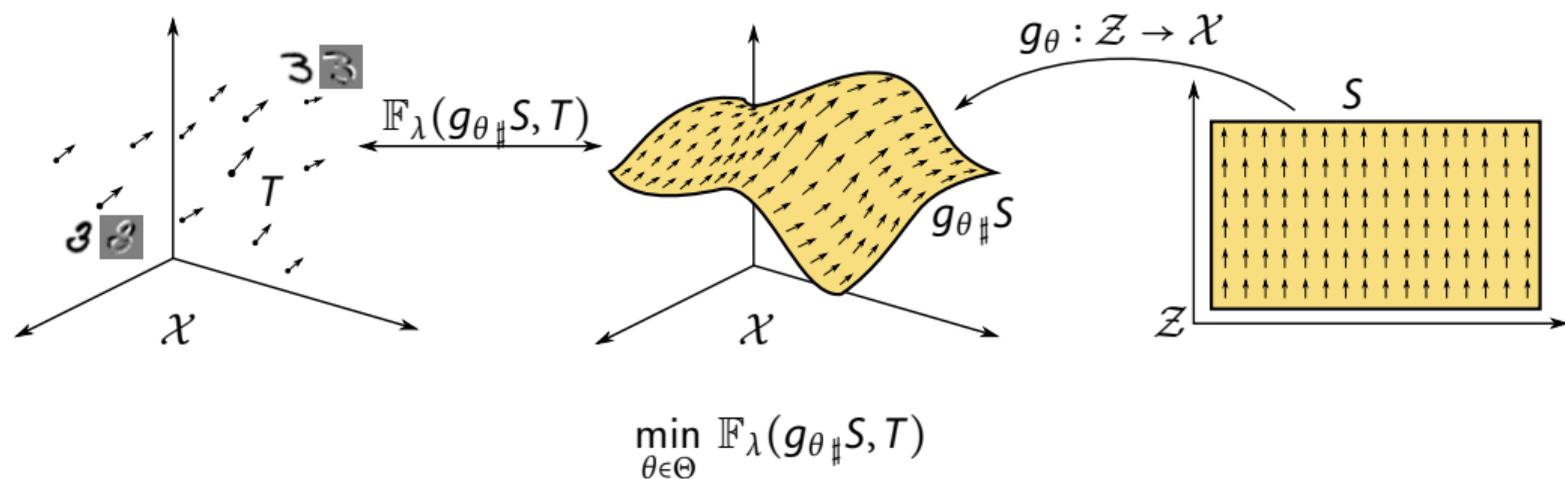


Federer & Fleming 1960: The flat metric metrizes the weak* convergence on normal currents with uniformly bounded mass and boundary mass.

Flat metric minimization: our theoretical result



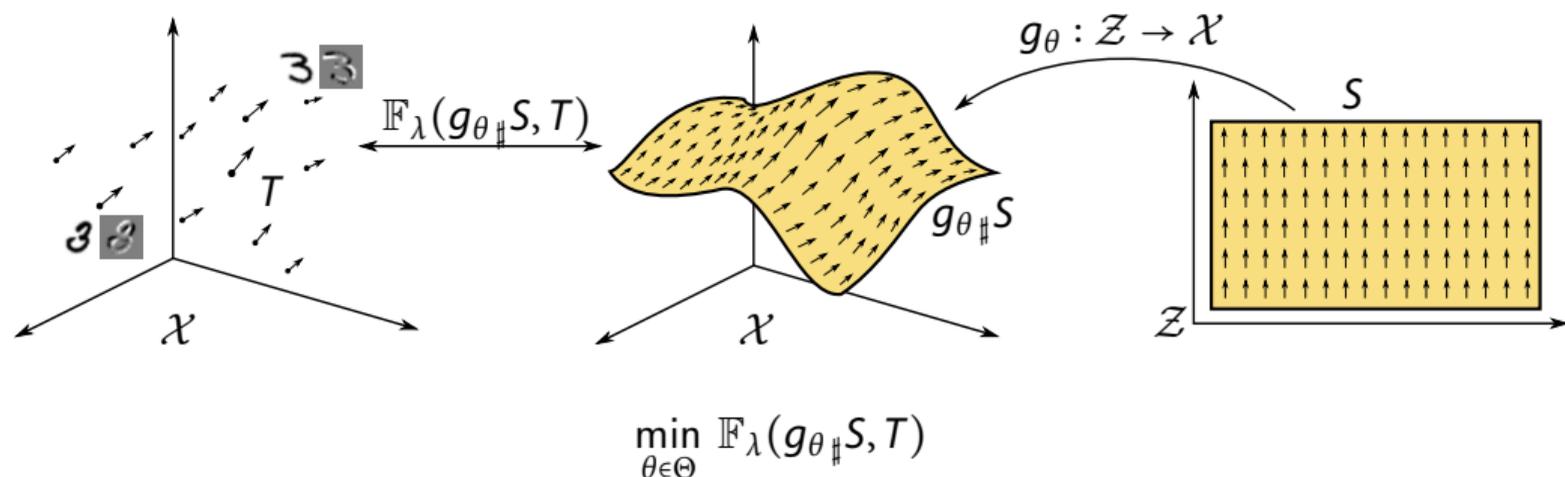
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Assumptions:

- ▶ Normal currents $S \in N_{k, \mathcal{Z}}(\mathbf{R}^l)$, $T \in N_{k, \mathcal{X}}(\mathbf{R}^d)$.
- ▶ $g : \mathcal{Z} \times \Theta \rightarrow \mathcal{X}$ smooth in z with uniformly bounded derivative, loc. Lipschitz in θ .
- ▶ Parameter space Θ is compact.

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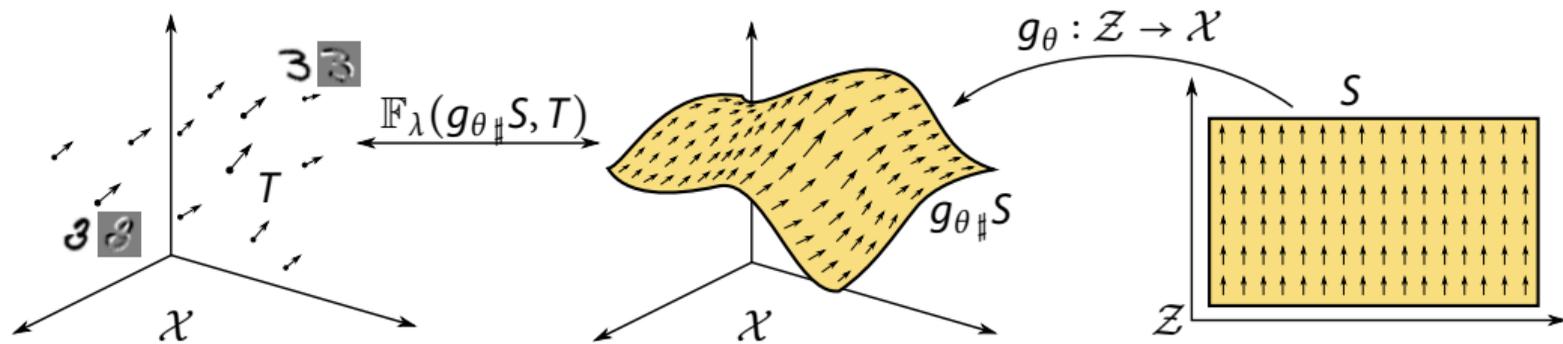


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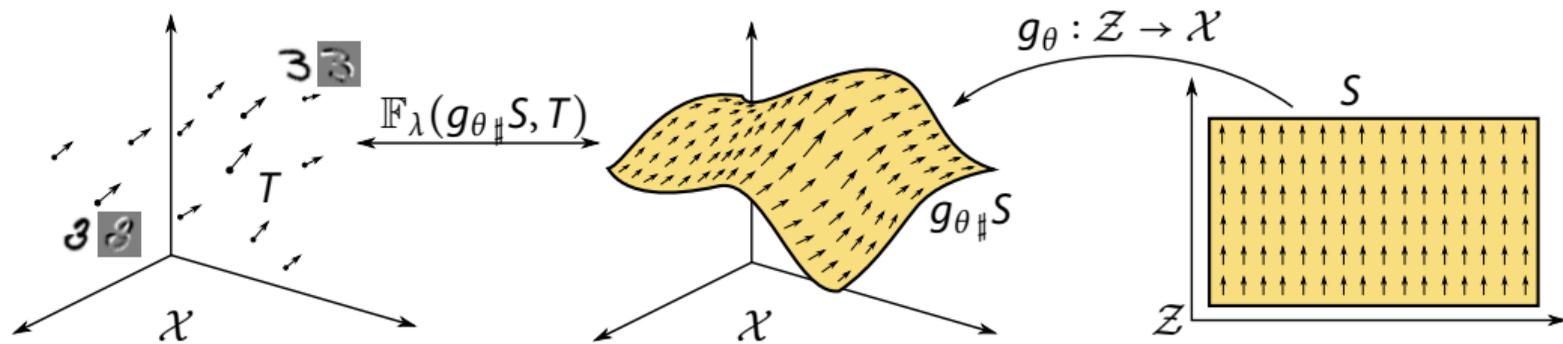
Proposition. The map $\theta \mapsto \mathbb{F}_{\lambda}(g_{\theta\#}S, T)$ is Lipschitz continuous.

FlatGAN formulation and implementation



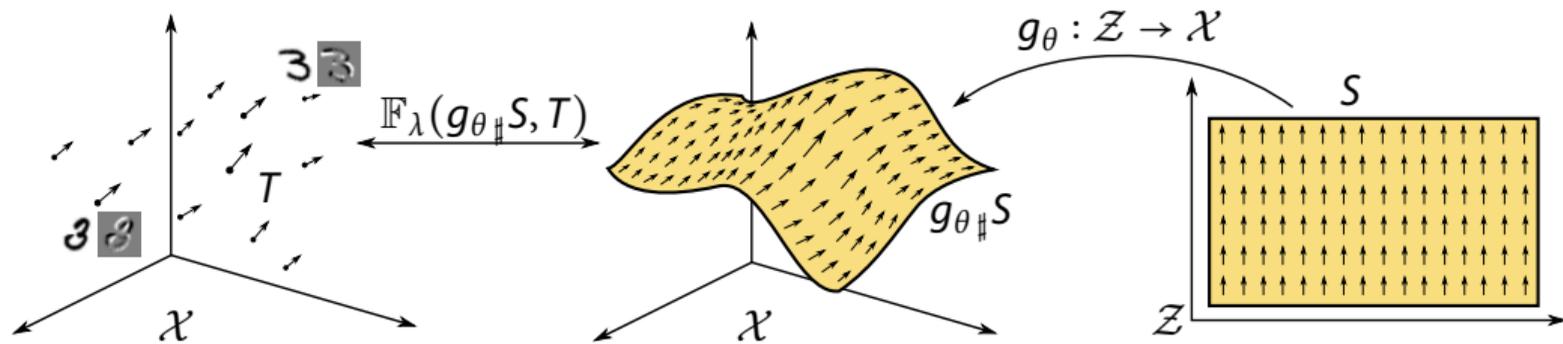
$$\min_{\theta \in \Theta} \mathbb{F}_\lambda(g_{\theta\#}S, T)$$

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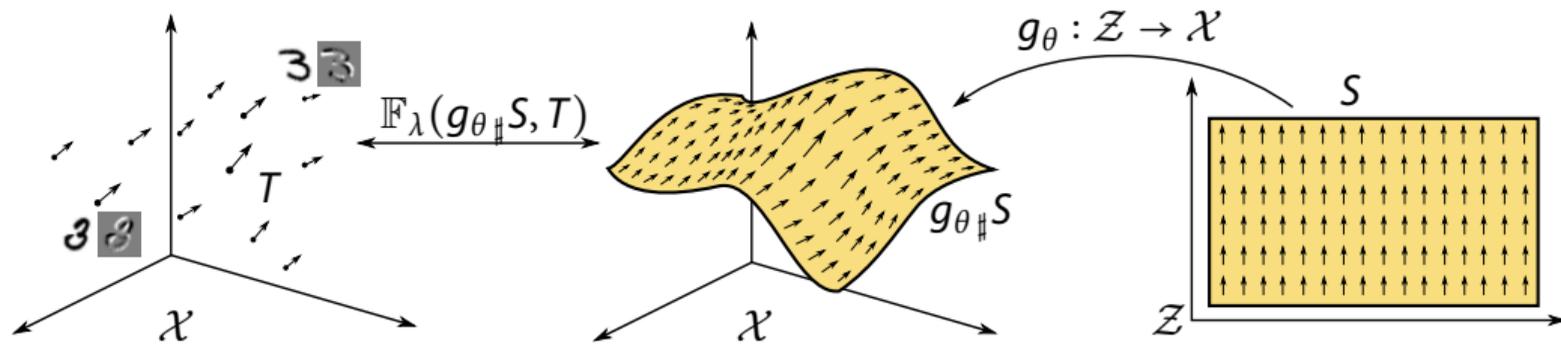
$$\min_{\theta \in \Theta} \sup_{\substack{\|\omega\|^* \leq \lambda \\ \|d\omega\|^* \leq 1}} (g_{\theta\#}S - T)(\omega)$$

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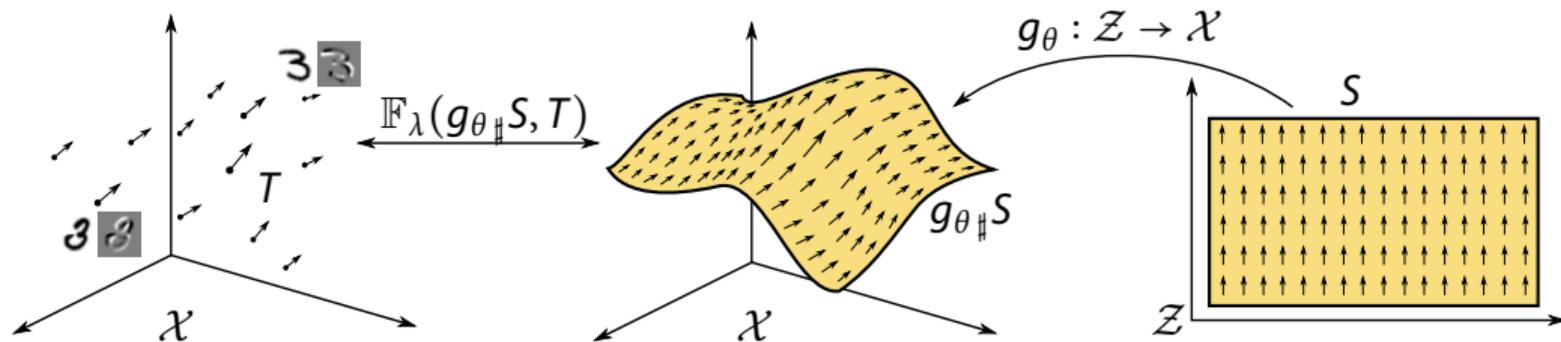
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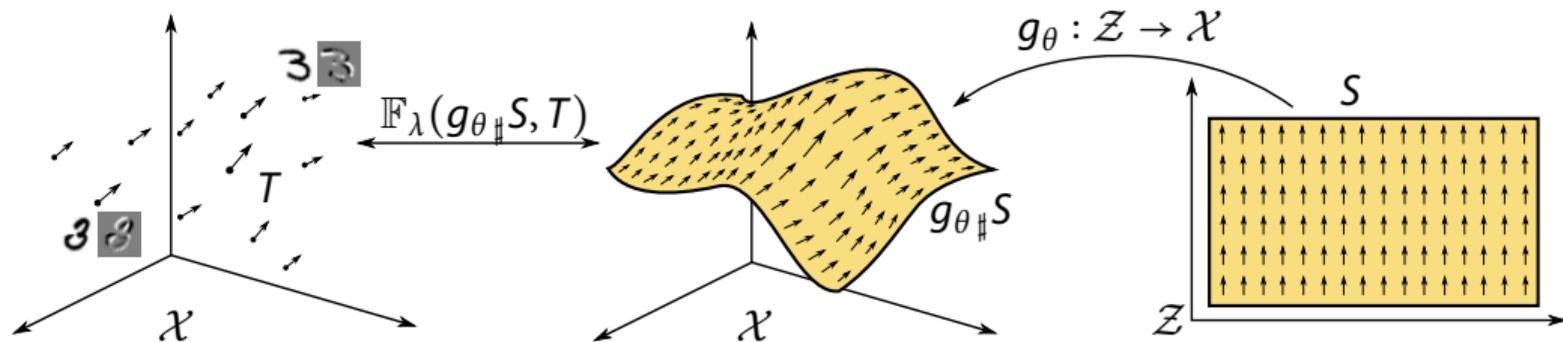
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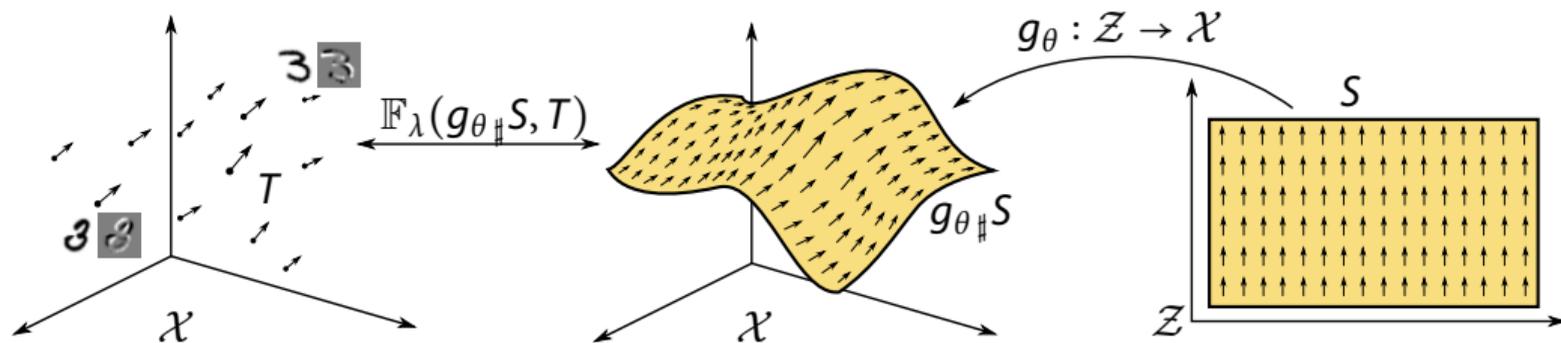
$$T = \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \wedge (T_{i,1} \wedge \dots \wedge T_{i,k}), S = \mu \wedge (e_1 \wedge \dots \wedge e_k)$$

FlatGAN formulation and implementation



$$\min_{\theta \in \Theta} \sup_{\substack{\|\omega\|^* \leq \lambda \\ \|d\omega\|^* \leq 1}} \mathbb{E}_{z \sim \mu} [\langle \omega \circ g_{\theta}, (\nabla_z g_{\theta} \cdot e_1) \wedge \cdots \wedge (\nabla_z g_{\theta} \cdot e_k) \rangle] - \frac{1}{N} \sum_{i=1}^N \langle \omega(x_i), T_{i,1} \wedge \cdots \wedge T_{i,k} \rangle$$

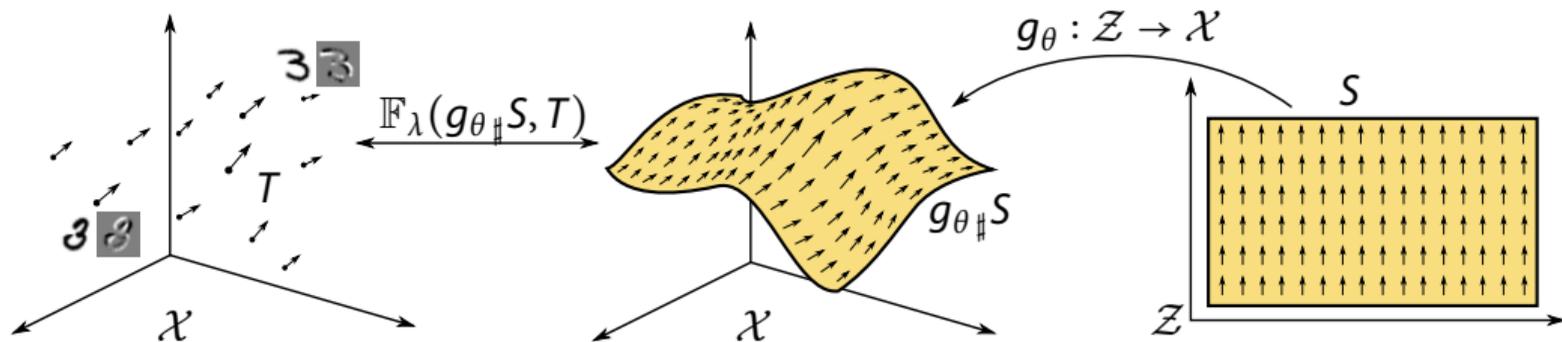
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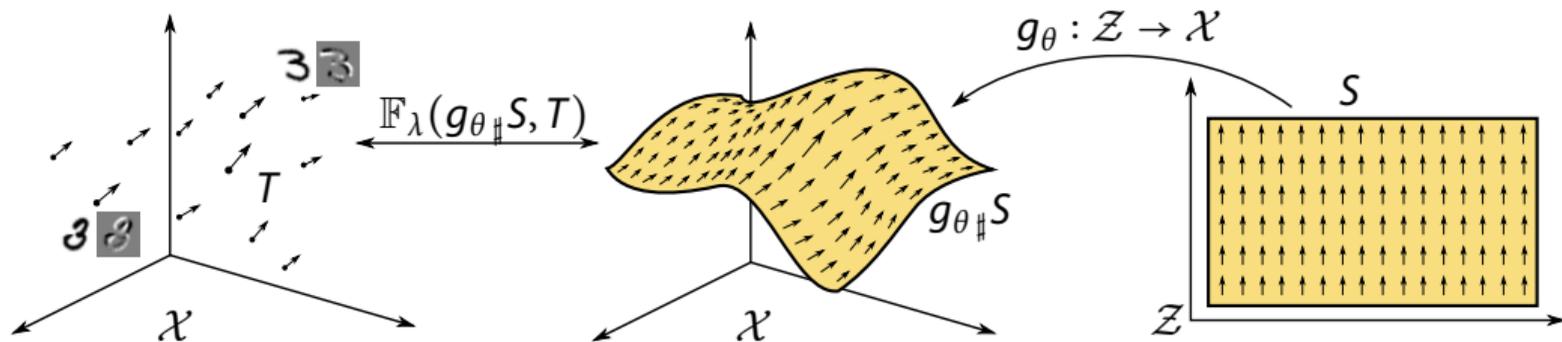
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- ▶ Implement $\omega : \mathbf{R}^d \rightarrow \Lambda^k \mathbf{R}^d$ and $g_{\theta} : \mathcal{Z} \rightarrow \mathcal{X}$ with deep nets
- ▶ Soft penalty for $\|\omega(x)\|^* \leq \lambda, \|d\omega(x)\|^* \leq 1$ (similar to WGAN-GP)

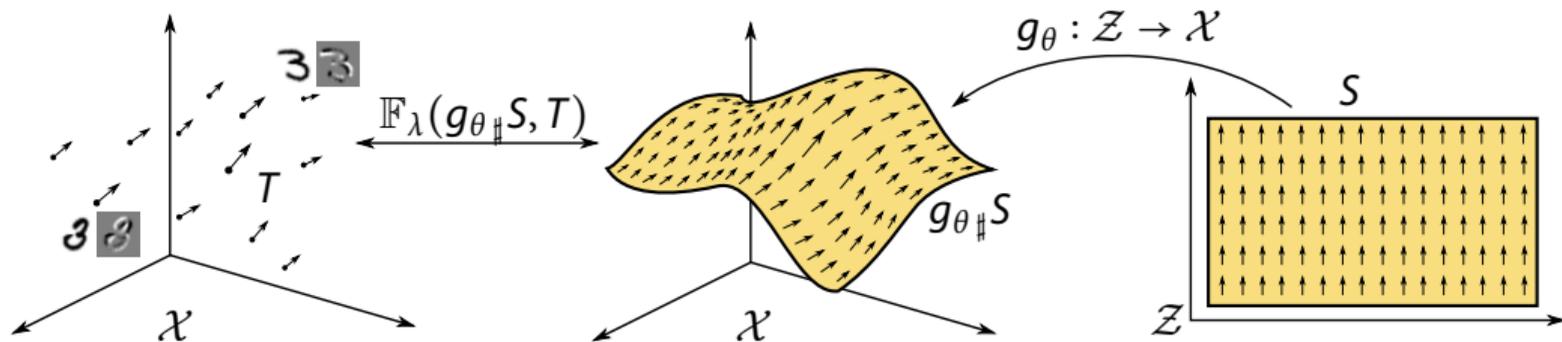
FlatGAN formulation and implementation



$$\min_{\theta \in \Theta} \sup_{\substack{\|\omega\|^* \leq \lambda \\ \|d\omega\|^* \leq 1}} \mathbb{E}_{z \sim \mu} [\langle \omega \circ g_{\theta}, (\nabla_z g_{\theta} \cdot e_1) \wedge \cdots \wedge (\nabla_z g_{\theta} \cdot e_k) \rangle] - \frac{1}{N} \sum_{i=1}^N \langle \omega(x_i), T_{i,1} \wedge \cdots \wedge T_{i,k} \rangle$$

- ▶ Implement $\omega : \mathbf{R}^d \rightarrow \Lambda^k \mathbf{R}^d$ and $g_{\theta} : \mathcal{Z} \rightarrow \mathcal{X}$ with deep nets
- ▶ Soft penalty for $\|\omega(x)\|^* \leq \lambda$, $\|d\omega(x)\|^* \leq 1$ (similar to WGAN-GP)
- ▶ Compute $\nabla_z g_{\theta} \cdot e_i$ with two calls to autograd (rop), $\langle \cdot, \cdot \rangle$ by $k \times k$ -determinant

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- ▶ Train model by alternating stochastic gradient ascent/descent

Illustration on a 2D toy data set (5 points on a circle)

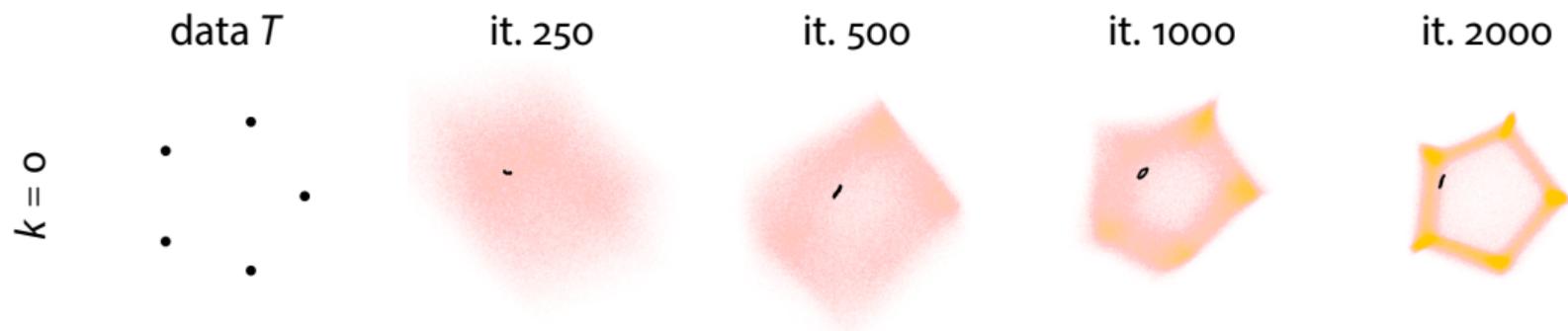
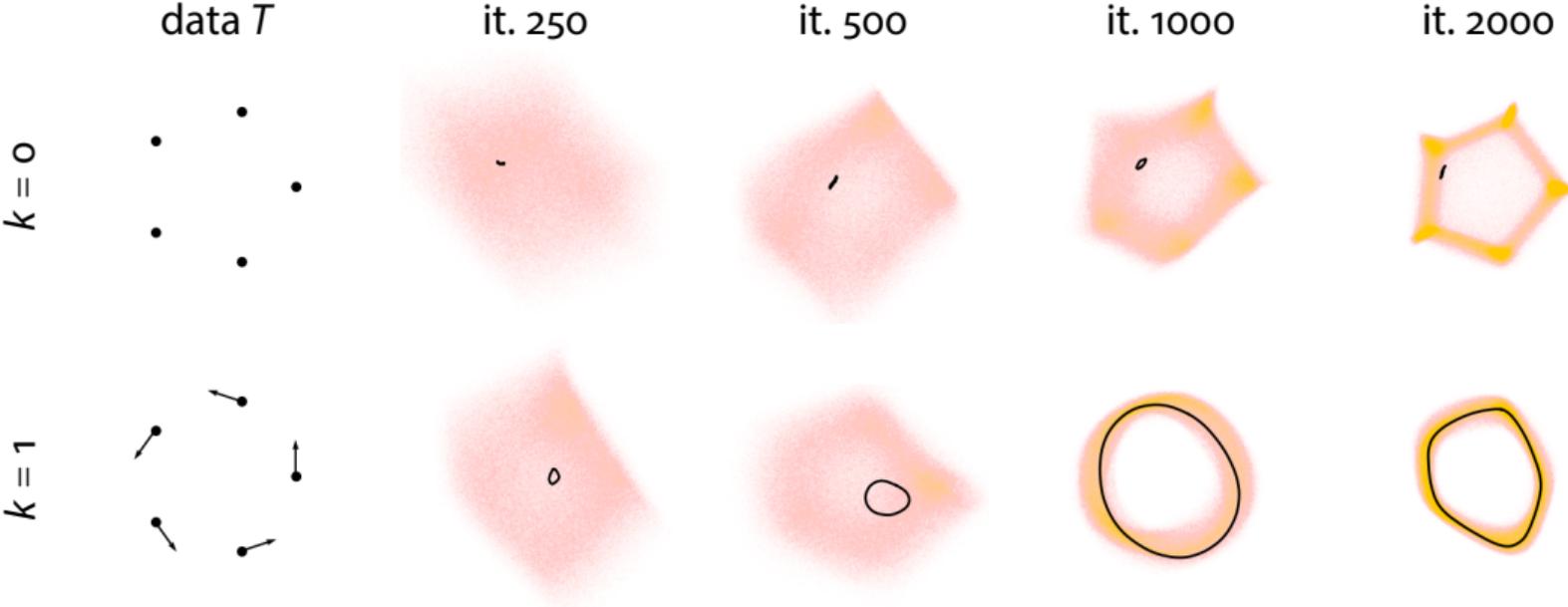


Illustration on a 2D toy data set (5 points on a circle)



Learning equivariant latent representations

MNIST, $k = 2$



varying z_1 (rotation)



varying z_2 (stroke width)

Learning equivariant latent representations

MNIST, $k = 2$

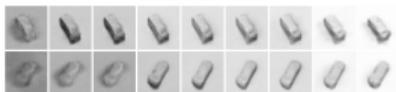


varying z_1 (rotation)



varying z_2 (stroke width)

smallNORB, $k = 3$



varying z_1 (lighting)



varying z_2 (elevation)



varying z_3 (azimuth)

Learning equivariant latent representations

MNIST, $k = 2$

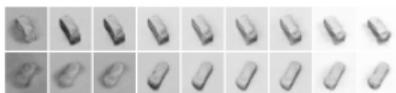


varying z_1 (rotation)



varying z_2 (stroke width)

smallNORB, $k = 3$



varying z_1 (lighting)



varying z_2 (elevation)



varying z_3 (azimuth)

tinyvideos, $k = 1$



varying z_1 (time)

See you at our poster, Pacific Ballroom #16, 6:30 tonight!



Flat Metric Minimization with Applications in Generative Modeling

Thomas Möllenhoff Daniel Cremers

Technical University of Munich



REPRESENTING DATA WITH NORMAL CURRENTS

Contribution: We propose to view (partially) oriented data as a k -current.

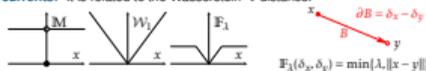


Intuitively, k -currents form a linear space that includes k -dimensional oriented manifolds as elements. The vector space of **normal currents** $N_{k,\chi}(\mathbb{R}^d)$ consists of currents T with finite volume and finite volume of their boundary: $M(T) + M(\partial T) < \infty$.

THE FLAT METRIC

$$F_1(S, T) = \min_{S-T = \partial B + A} M(B) + \lambda M(A) = \sup_{\substack{\|\omega\| \leq 1 \\ |\partial \omega| \leq 1}} S(\omega) - T(\omega)$$

For 0-currents: It is related to the Wasserstein-1 distance.



The intuition for 1-currents:



THEORETICAL RESULTS

Federer & Fleming 1960. The flat metric metrizes the weak* convergence on normal currents with uniformly bounded mass and boundary mass:

$$F_1(T, T_j) \rightarrow 0 \text{ if and only if } T_j \xrightarrow{*} T, \text{ i.e., } T_j(\omega) \rightarrow T(\omega), \text{ for all } \omega \in C_c^\infty(\mathbb{R}^d; \mathbb{A}^k \mathbb{R}^d).$$

Proposition. Let $S \in N_{k,\chi}(\mathbb{R}^d)$, $T \in N_{k,\chi}(\mathbb{R}^d)$ be normal currents. Assume $g_S: Z \rightarrow \mathbb{R}^d$ is smooth in z with uniformly bounded derivative and locally Lipschitz in θ . Then, the map $\theta \mapsto F_1(g_S S, T)$ is Lipschitz continuous on any compact parameter set Θ .

Presented at the International Conference on Machine Learning (ICML), Los Angeles, 2019.

FLATGAN: LEARNING EQUIVARIANT REPRESENTATIONS

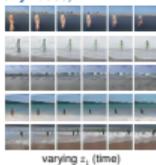
$$\min_{g \in \mathcal{G}} \left\{ F_1(g_S S, T) = \sup_{\substack{\|\omega\| \leq 1 \\ |\partial \omega| \leq 1}} \frac{1}{N} \sum_{i=1}^N (\omega(x_i), T_i) + \mathbb{E}_{z \sim p} \{ (\omega \circ g_S, (\nabla_z g_S \circ \epsilon_1) A \dots \wedge (\nabla_z g_S \circ \epsilon_k) B) \} \right.$$

Solving the above optimization problem yields a generator g_S which behaves equivariantly to the specified tangent vectors.

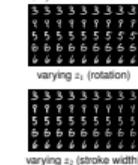
Illustration on a simple dataset in 2D:



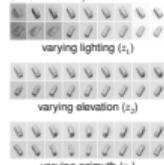
tinyvideos, $k=1$:



MNIST, $k=2$:



smallNORB, $k=3$:



GEOMETRIC MEASURE THEORY CHEAT SHEET & REFERENCES

- k -vectors and k -covectors.** $\mathbb{A}_k \mathbb{R}^d$ is a vector space in which some of the elements describe oriented, k -dimensional planes in \mathbb{R}^d . These are called **simple k -vectors**, $v_1 \wedge \dots \wedge v_k$. The dual space (k -covectors) is $\mathbb{A}^k \mathbb{R}^d$.
 - If v and w are simple, then we have $(v_1 \wedge \dots \wedge v_k, w_1 \wedge \dots \wedge w_k) = \det(Y^T W)$.
 - A differential form** is a k -covector field $\omega: \mathbb{R}^d \rightarrow \mathbb{A}^k \mathbb{R}^d$. **k -currents** are the dual space of smooth, compact k -forms.
 - $\int \omega = \int_{\text{supp}(\omega)} \langle \omega, v \rangle$. Area of the k -dim. parallelepiped spanned by the $\{v_i\} \in v_1, \dots, v_k$.
 - The mass $M(T) := \int \langle T, v \rangle$** is the k -dimensional volume of the k -current T .
 - Boundary:** $\partial T(\omega) = T(d\omega)$, where d is the exterior derivative (in \mathbb{R}^d : $\text{grad} \rightarrow \text{curl} \rightarrow \text{div}$)
 - Orientations:** Continuous k -vector map $\nu: M \rightarrow \mathbb{A}^k \mathbb{R}^d$, $\nu(x)$ is simple with unit norm, spanning $T_x M$ for all $x \in M$.
 - Stokes' theorem:** $\int_{\partial M} \langle \omega, \nu \rangle d\mathcal{H}^k = \int_M \langle d\omega, \nu \rangle d\mathcal{H}^k$, it follows that $\partial \langle M, \omega \rangle = \langle \partial M, \omega \rangle$.
 - Pushback:** $(g^* \omega, \nu_1 \wedge \dots \wedge \nu_k) = (\omega \circ g, \nabla_{z_1} \nu_1 \wedge \dots \wedge \nabla_{z_k} \nu_k)$. **pushforward:** $g_* T(\omega) = T(g^* \omega)$.
- [1] H. Federer and W. H. Fleming. Normal and integral currents. *Annals of Mathematics*, pages 458–520, 1960.
 [2] H. Federer. *Geometric Measure Theory*. Springer, 1969.
 [3] F. Morgan. *Geometric Measure Theory: A Beginner's Guide*. Academic Press, 5th edition, 2016.

PyTorch implementation: <https://github.com/moellenh/flatgan>